MATH 239 - S23
Introduction to Combinatorics
Full Course Notes

With Prof Logan Crew

My notes consist of most definitions, and some summaries, methods, and examples. They are not exhaustive.

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$\mathbb{N}$ includes 0

1 Definitions

- $\{\}$ **Set**: collection of distinct objects.
- $\in$ **Element**: object within a set.
- $\subseteq$ **Subset**: all items are in parent set.
- $|S|$ **Cardinality**: number of elements in the set.
- $\cup$ **Union**: all objects in either set.
- $\cap$ **Intersection**: all objects in both sets.
- $\setminus$ **Deletion**: in first, not in second.
- $\times$ **Cartesian Product**: set of ordered pairs, from each.
- $\emptyset$ **Empty Set**: it's unique. $\emptyset \subseteq$ any set

Sizes

$$|S \cup T| = |S| + |T| - |S \cap T|$$

$$|S \setminus T| = |S| - |S \cap T|$$

$$|S \times T| = |S| \cdot |T|$$

$$\left|\text{all subsets of } S\right| = 2^{|S|} \leftarrow \text{Powerset}$$

$$\left|\text{k-element subsets of } S\right| = \binom{|S|}{k}$$

2 Bijections, Proofs

Bijection: a map $f: A \to B$ that is 1. injective 2. surjective

Injective: $f(a_1) = f(a_2) \to a_1 = a_2$

Surjective: $\forall b \in B, \exists a \in A \mid f(a) = b$

Indicator Variable: value that represents a choice

Tuple: Ordered set.

★ **Proof of Bijection** ↳ "same size" proof

1. Show is injective & surjective
2. Define f, f^{-1} and show $f(f^{-1}(a)) = f^{-1}(f(a)) = a$

tips

- A: Clearly define your sets; give them names
- B: Use words in your definitions freely
- C: Illustrate expected inputs & outputs

Finite sets with a bijection between them have the same size.

Proof Example

Q: show $|P(T)| = 2^{|T|}$

A: Consider the map $f: P(T) \to \{0,1\}^{|T|}$ then $|\{0,1\}^{|T|}| = 2^{|T|}$.

Let $T = \{t_1, \dots, t_{|T|}\}$...

Given $S \subseteq T$,

$$f(S) = \begin{cases} i\text{th coor} = 0 \text{ if } t_i \notin S \\ i\text{th coor} = 1 \text{ if } t_i \in S \end{cases}$$

Then f is a bijection...

To show, we define f^{-1}:

$$f^{-1}(s) = \begin{cases} \text{if } t_i = 0, \text{ don't put } S_i \text{ in } S \\ \text{if } t_i = 1, \text{ put } S_i \text{ in } S \end{cases}$$

and show $f^{-1}(f(a)) = a$ and $f(f^{-1}(a)) = a$

3 Combinatorial Proofs, Multisets

Permutation: an ordering of some elements.

List: ordered set (a specific permutation).

★ **Combinatorial Proof**

1. Come up with sets and "just start."
2. Create the easiest bijection you can.
3. Define bijection! eg: $f(T) = \begin{cases} T & \text{if } T \in S_{n-1,k} \\ T \cup \{n\} & \text{if } T \in S_{n-1,k-1} \end{cases} \leftarrow f: A \cup B \to C$

- A: If you have $f: A \cup B \to C$, show $f^{-1}: C \to A$ or B specifically.
- B: When making sets on same side, make them disjoint.

Multiset: ... of size n with t types is a sequence (tuple) of non-negative integers $m_1, \dots, m_t$ s.t. $m_1 + \dots + m_t = n$

BINOMIAL COEFFICIE

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

$$\left(\!\binom{n}{t}\!\right) = \binom{n+t-1}{n}$$

$$\sum_{i=1}^{n} i = \binom{n+1}{2}$$

4 Combinatorial Intuitions

Inclusion/Exclusion:

$\left(\!\binom{n}{t}\!\right)$ # of ways to choose n items from $[t]$ with repetition!

5 Power Series (formal)

- $G(x) = \sum_{n=0}^{\infty} c_n x^n$
- $G(x) + H(x) = \sum_{n=0}^{\infty} (c_n + d_n) x^n$
- $G(x)H(x) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} c_i d_{n-i} \right) x^n$

NEVER evaluate a power series at a real number.

Tips
1. Factor
2. Subtract

$[x^n]G(x) \to$ coefficient on x^n

$(1 + x + x^2 + \dots)(1 - x) = 1$

$\frac{1}{1-x}$ is the Geometric Series

eg:

$$[x^{10}] \frac{3x}{7+2x} = [x^9] \frac{3}{7+2x} = [x^9] \frac{3}{7} \cdot \frac{1}{1 + \frac{2}{7}x} = \frac{3}{7}[x^9] \frac{1}{1 - (-\frac{2}{7}x)} = \frac{3}{7}[x^9] \sum_{i=0}^{\infty} \left(-\frac{2}{7}x\right)^i = \frac{3}{7} \cdot \left(-\frac{2}{7}\right)^9$$

Find out if $G(x)$ has inverse

$$\frac{1}{G(x)} \underset{\text{rewrite so it looks like...}}{=} \frac{1}{1 - (1 - G(x))} = \sum_{i=0}^{\infty} (1 - G(x))^i$$

If $[x^0]G(x) = 0$, then $G^{-1}(x)$ isn't well-defined

$$[x^k] \sum_{n=0}^{\infty} (1 - G(x))^n = \sum_{n=0}^{k} [x^k] (1 - G(x))^n$$

Generating Function for A

Weight fn $w: A \to \mathbb{N} = \{0, 1, \dots\}$

$$\Phi_A^w(x) = \sum_{a \in A} x^{w(a)}$$ ★

$[x^k]\Phi_A^w(x) =$ # of objects in A w/ weight k

EG $w: \mathbb{N} \to \mathbb{N}$, $w(n) = n$, $\Phi_{\mathbb{N}}^w = \frac{1}{1-x}$

★ Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

6) More Weight Functions

- Weight functions give finitely many elements of each weight.

Negative Binomial Theorem

$$(1-x)^{-t} = \sum_{n=0}^{\infty} \binom{n+t-1}{n} x^n$$

7) More Generating Series

Sum Lemma

$$\Phi^{w}_{A\cup B}(x) = \Phi^{w_A}_{A}(x) + \Phi^{w_B}_{B}(x)$$

IF $A \cap B = \emptyset$

Product Lemma

$$\Phi^{w}_{A\times B}(x) = \Phi^{w_A}_{A}(x)\,\Phi^{w_B}_{B}(x)$$

String Lemma

As long as $0 \notin w_D$:

$$\Phi^{w_{D^*}}_{D^*}(x) = \sum_{k=0}^{\infty}\left(\Phi^{w_D}_{D}\right)^k = \frac{1}{1-\Phi^{w_D}_{D}(x)}$$

Eg: Cartesian stuff

Let (A, w_A), (B, w_B).
set, weight function

$$\Phi^{w_{A\times B}}_{A\times B} = \sum_{(a,b)\in A\times B} x^{w_{A\times B}(a,b)}$$
$$= \sum_{(a,b)\in A\times B} x^{w_A(a)+w_B(b)}$$
$$= \Phi^{w_A}_{A}(x)\,\Phi^{w_B}_{B}(x)$$

EG: Dice Rolls

$D = \{1, \dots, 6\}$

$w_D(d) = d$

$$\Phi^{w_D}_{D}(x) = x+x^2+x^3+x^4+x^5+x^6$$
$$= x\,\frac{1-x^6}{1-x}$$
$$= \frac{x-x^7}{1-x}$$

3 rolls

$$\Phi^{w_{D^3}}_{D^3}(x) = \left(\Phi^{w_D}_{D}(x)\right)^3$$

eg: Weight-preserving bijection

$$f: S_n \to \{0,1\}^n$$

set of subsets of $\{1, \dots, n\}$; n-tuples containing 0 or 1

where

$$W_{S_n}(A) = |A|,\quad A \in S_n$$

$$\Phi_{S_n}(x) = \sum_{k=0}^{n}\binom{n}{k}x^k$$

$$\Phi_{\{0,1\}^n}(x) = (1+x)^n$$

choice

EG: Rolling N Dice

$$D^0 \cup D^1 \cup D^2 \cup \dots = D^*$$
$$= \Phi^{w_{D^0}}_{D^0}(x) + \Phi^{w_{D^1}}_{D^1}(x) + \Phi^{w_{D^2}}_{D^2}(x) + \dots$$
$$= \sum_{k=0}^{\infty}\left(\Phi^{w_D}_{D}\right)^k = \frac{1}{1-\Phi^{w_D}_{D}(x)}$$

Restrictions: $0 \notin w_D$ because then w_{D^*} would have infinite constant

eg: Generating Series

w.f. $w: A \to \mathbb{N}$

Its generating series:

$$\Phi^{w}_{A}(x) = \sum_{a\in A} x^{w(A)}$$

Suppose $f: A \to B$ is a bijection that preserves weights. Then:

$$\Phi^{w_a}_{A}(x) = \Phi^{w_b}_{A}(x)$$

eg: Algebra

$$(3+2x)^7 = 3^7\left(1+\tfrac{2}{3}x\right)$$
$$= 3^7 \sum_{k=0}^{7}\binom{7}{k}\left(\tfrac{2}{3}x\right)^k$$

EG: Generating Function

Write generating function for all finite-length binary strings with weight equal to length.

Consider length 1.

$w_B(0) = w_B(1) = 1$

$B = \{0, 1\}$

$$\Phi^{w_B}_{B}(x) = \sum_{b\in B} x^{w_B(b)} = \boxed{2x}$$

8) Compositions

Composition: finite length tuple of positive integers.

$(2, 8, 10, 1) = \alpha$ $\quad \ell(\alpha) = 4$

1st part

SIZE $\to |\alpha| = 21$

empty: ε

$\ell(\varepsilon) = 0$

$|\varepsilon| = 0$

Compositions weighted by size: Generating Function

$$\Phi^{w_{\mathbb{N}^*_{>0}}}_{\mathbb{N}^*_{>0}}(x) = \frac{1-x}{1-2x}$$

parts must be positive

9) Compositions (again)

Combinatorial Interpretation: Write n 1's, then partition. That will be n-1 positions! 2^{n-1} choices

Method: ① Generating series for R={restriction} ② Map to series for R^*!

EG:

$$[x^n]\,\frac{1-x}{1-2x}\ ?$$
$$= [x^n]\,(1-x)(1-2x)^{-1}$$
$$= [x^n](1-2x)^{-1} - [x^n]\,x(1-2x)^{-1}$$
$$= 2^n - [x^{n-1}](1-2x)^{-1}$$

by geometric series

$$= 2^n - 2^{n-1}$$
$$= 2^{n-1} \text{ if } n>0,\ 1 \text{ otherwise}$$

10) Strings

Substr: of $S = s_1 \cdots s_n$ is $s_i \cdots s_j$ with $1 \le i \le j \le n$

(sum) **Concatenation:** of $s, t \in \{0,1\}^*$ is st.

(prod) **Concatenation Product:** of $S, T \subseteq \{0,1\}^*$ is $ST = \{st : (s,t) \in S \times T\}$

(string) **All strings:** $S^* = \varepsilon \cup S \cup S^2 \dots = \sum_{k=0}^{\infty} S^k$ ← "concatenation star"

Block: maximal substring of an individual letter.

Block decompositions: are unique! $S = \{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$ — only for binary

Unambiguous: when each result can only be made one way

★ for Φ_{ST} ★

$\Phi_{ST} = \Phi_S(x)\,\Phi_T(x)$ iff there is a bijection between $S\times T \leftrightarrow ST$

(implies)

EG: How many compositions of 123 where each part is 1 or 3?

Let $R = \{1, 3\}$. Then $\Phi_R(x) = x^1 + x^3$.

Set of all compositions with parts 1 or 3 is R^*.

We want $[x^{123}]\,\Phi_{R^*}(x)$

$$= [x^{123}]\,\frac{1}{1-x-x^3}$$

Hockey Stick Identity

$$\binom{a+b}{a} = \sum_{i=0}^{b}\binom{a+i-1}{a-1}$$

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof excercise:

The number of compositions of n-1 using only odd parts is the same as # of n compositions using only parts ≥ 2.

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11 RegEx, GS, Block Decomposition

Regex:
- $\varepsilon, 0, 1$ are all regexes.
- $R \overset{\text{cup}}{\smile} S$, RS, R^*, R^i

Interpret

$\varepsilon \to \{\varepsilon\}$

$0 \to \{0\}$

$1 \to \{1\}$

$R \smile S \to R \cup S$

$RS \to RS$

$R^* \to R^*$

EG

$\{0,1\}^*$ is $(0 \smile 1)^*$

$(0 \smile 10)^*$ is all binary strings where each 1 is followed by a 0

$(00 \smile 1)^*$ is all binary strings where each 0 block has even length

Make GS

ε	leads to	1
0	–"–	x
1	–"–	x
$R \smile S$	–"–	$\Phi_R + \Phi_S$
RS	–"–	$\Phi_R \Phi_S$
R^*	–"–	$\dfrac{1}{1-\Phi_R}$

if $R \cap S = \emptyset$ (unambiguous)

$RS \rightleftarrows R \times S$

if $\bigcup_{i \geq 0} R^i$ is unambiguous

EG

0^* is unambiguous

$(0 \smile 11)^* \smile (00 \smile 1)^*$ is ambiguous

$1^*(01^*)^*$ is unambiguous

$1^*(0(00)^*11^*)^*(0(00)^* \smile \varepsilon)$ is un?

Solving Word problems

★ ① Rephrase into positive ② Write the regex ②.⑤ Use slices/blocks ③ Get gs, extract n^{th} coeff

12 Decompositions, Coefficients

EG # of strings of length n without 111 as a substring?

Standard Decomp: $S = 0^*(11^*00^*)^*1^*$

Change to $R = 0^*((1 \smile 11)00^*)^*(\varepsilon \smile 1 \smile 11)$

★ then compute g.s. ★

13 Block Decompositions → Generating Series

14 Recurrence Relations

EG Coefficient Extraction from $\dfrac{1+x+x^2}{1-x-x^2-x^3} = F(x)$

$F(x) = \sum_{n \geq 0} f_n x^n$ what we're tryna solve for!

$\Rightarrow 1 + x^2 + x = (1 - x - x^2 - x^3) \sum_{n\geq 0} f_n x^n$

$= \sum_{n\geq 0} f_n x^n - \sum_{n \geq 0} f_n x^{n+1} - \sum_{n\geq 0} f_n x^{n+2} - \sum_{n\geq 0} f_n x^{n+3}$

$= \sum_{n\geq 0} f_n x^n - \sum_{n\geq 1} f_n x^n - \sum_{n\geq 2} f_n x^n - \sum_{n\geq 3} f_n x^n$

$1x^0 + 1x^1 + 1x^2 = f_0 x^0 + f_1 x^1 - f_0 x^1 + f_2 x^2 - f_1 x^2 - f_0 x^2 + \sum_{n \geq 3} x^n (f_n - f_{n-1} - f_{n-2} - f_{n-3})$

$= f_0 x^0 + (f_1 - f_0) x^1 + (f_2 - f_1 - f_0) x^2 + \text{sum}$

	LHS		RHS
x^0	1	=	$f_0 \to f_0 = 1$
x^1	1	=	$f_1 - f_0 \to f_1 = 2$
x^2	1	=	$f_2 - f_1 - f_0 \to f_2 = 4$
$x^n (n \geq 3)$	0	=	$f_n - f_{n-1} - f_{n-2} - f_{n-3}$

initial conditions

$f_n = f_{n-1} + f_{n-2} + f_{n-3}$

recurrence relations!

15 Recursive Decompositions + Forbidden Substrings

Recursive Decomposition: a regex w/ a "variable" on both sides.

$S = \varepsilon \smile (0 \smile 1)S \longrightarrow \Phi_S(x) = 1 + (x + x)\Phi_S(x)$

$\Phi_S(x) - 2x\Phi_S(x) = 1$

$\Phi_S(x) = \dfrac{1}{1-2x}$

Substrings

κ is a substring of σ if $\exists \alpha, \beta \in \{0,1\}^*$ s.t. $\alpha\kappa\beta = \sigma$

★ System of Equations!

EG Forbidden Substrings

Let A be the set of strings that don't contain κ.
Let B be the set of strings with exactly one κ, at the end.

Q Find the generating series of A and B

A Get two equations that relate A and B.

Equation 1: Adding one bit to A either creates an instance of κ at the end or doesn't. Also, A & B are disjoint. So we have:

$A(0 \smile 1) = A \cup B \to A(x)(x+x) = A(x)B(x)$

$\to B(x) = (2x-1)A(x)$

Equation 2: $A\kappa$ is similar to B, but κ may overlap with itself, so this relation depends on κ.
Suppose $\kappa = 01101$. Then an overlap is $(011(01)101)$, giving:

$A(01101) = B \cup B(101) \to A(x)x^5 = (1 + x^3)B(x)$

then just algebra to finish solving the question

16 Linear Recurrences & Partial Fractions

To get an exact formula by partial fractions, Factor The Denominator. So for $\dfrac{P(x)}{Q(x)}$,

When Does This Work? ① $P(x)$ doesn't matter ② Can do if $Q(x) = (1-cx)^t$ NBT ③ Get partial fractions

We want $Q(x) = \prod(1 - r_i x)$, not $\prod(x - r_i)$...

Reverse coefficients $x^5 + x^2 \to 1 + x^3$ → Solve for the roots

EG

$\dfrac{1}{1-x-x^2} \to C_n - C_{n-1} - C_{n-2} = 0$

EG Partial Fractions

$Q(x) = 1 - x - x^2$

constant term of 1: Done

reverse coefficients: $x^2 - x - 1$

find roots: $r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$

partial fraction decomposition: $\frac{1}{1-x-x^2} = \frac{A}{1-r_1x} + \frac{B}{1-r_2x} = \frac{A+B-r_2Ax-r_1Bx}{1-x-x^2}$

$A+B=1$, $-r_2A - r_1B = 0$ $\Rightarrow$ $\frac{5+\sqrt{5}}{10}\cdot\frac{1}{1-r_1x} + \frac{5-\sqrt{5}}{10}\cdot\frac{1}{1-r_2x}$

By the NBT

Get coefficients: $[x^n]\frac{1}{1-x-x^2} = \frac{5+\sqrt{5}}{10}r_1^n + \frac{5-\sqrt{5}}{10}r_2^n$

PARTIAL FRACTIONS

Suppose $Q(x) = \prod_{i=1}^{t}(1-r_ix)^{\lambda_i}$.

Suppose $\deg P(x) < \deg Q(x)$.

Then $\frac{P(x)}{Q(x)} = \sum_{i=1}^{t}\sum_{j=1}^{\lambda_i}\frac{A_i^{(j)}}{(1-r_ix)^j}$

EXAMPLE

$\frac{1}{(1-x)(1-2x)^2} = \frac{A_1^{(1)}}{1-x} + \frac{A_2^{(1)}}{1-2x} + \frac{A_2^{(2)}}{(1-2x)^2}$

Goodbye Comby, Hello Graphy!

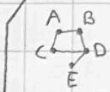
(17) Graph Theory Notation

$V(G)$ is never $\{\}$

Graph: A set of vertices $V(G)$ and a set of edges $E(G)$.

Adjacent: The edge AB connects (is incident with) A and B (neighbours), the endpoints.

eg Graph H



$V(H) = \{A, B, C, D, E\}$

$E(H) = \{AB, AC, BD, CD, DE\}$, $\{C,D\} \leftarrow CD$

Isomorphic: H and J are isomorphic if $\exists$ function $f: V(H) \to V(J)$ that is bijective that satisfies that if $b_1, b_2 \in V(H)$, $b_1b_2 \in E(H)$ iff $f(b_1)f(b_2) \in E(J)$. Then f is an isomorphism.

In this class, any result will hold "up to isomorphism."

Automorphism: an isomorphism of a graph with itself.

eg Let $p: V(H) \to V(H)$

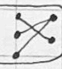
$p(A)=A$, $p(B)=C$, $p(C)=B$, $p(D)=D$, $p(E)=E$

p is a non-trivial automorphism of H

(18) New Graps

Complete Graph $K_5 =$ 

All possible edges. $|E(K_n)| = \frac{n(n-1)}{2}$

Bipartite Graph: eg 

$V(G)$ can be split into A & B s.t. $\forall e \in E(G)$ has 1 endpoint in A & B.

Complete Bipartite: eg  $K_{3,2}$

Has vertices $a_1, \dots, a_m, b_1, \dots, b_n$, and edges a_ib_j for $1 \le i \le m$, $1 \le j \le n$. $|E(K_{m,n})| = m \times n$

Neighbourhood of $v \in V(G)$ is the set of vertices adjacent to v. Denoted $\boxed{N(v)}$

Degree $d(v) = |N(v)|$

Handshaking Lemma

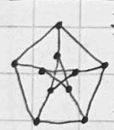
$$\sum_{v \in V(G)} d(v) = 2|E(G)|$$

Parity Lemma

The number of vertices of odd degree in a graph is even.

K-regular if every vertex has degree k.

Peterson Graph



3-regular, triangle-free, symmetric

(19) Paths, Cycles

GIVE NAMES

Walk: start/ends on a vertex; it's a string $V_0, V_0V_1, V_1, V_1V_2 \dots V_n$.

Closed Walk: starts/ends on same vertex.

Length: # of edges

Path: walk with no repeated vertices.

Cycle: a closed walk with at least 3 distinct vertices where all vertices are distinct except the first and last.

Path Lemma

If there is a walk from x to y, there is a path from x to y.

Writing Proofs

- Name things
- Be specific about your proof techique.

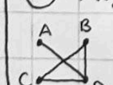
(20) Graphs, Subgraphs

Lemma

If every vertex of a graph is of degree 2, then the graph has a cycle.

How to Specify a Graph

1. Define $V(G)$ and $E(G)$
2. Draw and label it
3. Adjacency Matrix



$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 0 & 0 & 1 \\ B & 0 & 0 & 1 & 1 \\ C & 0 & 1 & 0 & 1 \\ D & 1 & 1 & 1 & 0 \end{array}$$

$A_{ii} = 0$, $A_{ij} = A_{ji}$

eg $A(G) \times \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ $|V(G)|$ times gives us a vector $\begin{bmatrix} \deg(V_1) \\ \deg(V_n) \end{bmatrix}$

Subgraph: Given graph G, a subgraph H is a graph s.t. $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Clique: Subgraph isomorphic to K_n.

Stable Set: subgraph isomorphic to empty graph.

Spanning Subgraph: $V(H) = V(G)$

Hamilton Path: A path containing all vertices

Hamilton Cycle: A cycle containg all vertices

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21 Connectivity & Components

Connected: G is connected if there is a path between every pair of vertices.

Component: a maximal connected subgraph.
$G_1 = (V(G_1), E(G_1))$ is a component of G if whenever F is a subgraph of G with either $V(G_1) \subsetneq V(F)$ or $V(G_1) = V(F)$ and $E(G_1) \subsetneq E(F)$ then F is disconnected.

Isolated Vertices: are components.

Component Decomposition Lemma

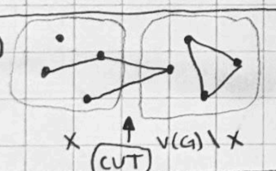
$V(G)$ can be written uniquely in some way $V(G_1) \cup V(G_2) \cup \dots \cup V(G_k)$ s.t. the subgraph $G_i = (V(G_i)$, all edges in $G_i)$ is a component.

- Each vertex is in exactly 1 component.
- G is connected iff it has exactly 1 component.
- No edges exist between distinct components.
- Each edge is in exactly 1 component.

Cut of X in G

Let G be a graph, and $X \subseteq V(G)$.
Cut of X in G: $\delta_G(X)$ = set of edges with one endpoint in X and one endpoint in $V(G) \setminus X$.

eg



$|\delta_G(X)| = 2$

Connectivity Theorem

G is connected iff $\forall X \subseteq V(G)$ s.t. $X \neq \emptyset$ and $X \neq V(G)$, the cut of X in G is non-empty.

22 Bridges

Delete: if $v \in V(G)$, $G \setminus v = (V \setminus v,\ E \setminus \delta_G(\{v\}))$
if $e \in E(G)$, $G \setminus e = (V, E \setminus e)$

Bridge: an edge s.t. $G \setminus e$ has more components (than G).

Bridge Delete Lemma

Let G be a graph with $x, y \in V(G)$, $xy \in E(G)$. Suppose G is connected, and xy is a bridge.
Then $G \setminus xy$:
(a) Has exactly 2 components.
(b) x and y are in diff components.

23 More Bridges

Bridge Cycle Lemma

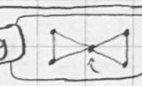
An edge is a bridge iff it is not in any cycle

Cut Vertex is a vertex v s.t. $G \setminus v$ has more components than G.

Cut Vertex Lemma

If G is a connected graph w/at least 3 vertices. Let e be a bridge of G. Then at least one endpoint of e is a cut vertex of G.

- You can have cut vertices without bridges. eg



Eulerian Circuit	closed walk on every edge exactly once

24 Eulerian Circuits

Eulerian Circuit: closed walk, on every edge exactly once.

Eulerian Theorem

A connected graph G has an Eulerian circuit iff all vertices have even degree.

25 Trees and Forests

Forest: Graph where every edge is a bridge.
Tree: a connected forest.
Forest: disjoint union of trees.

For a connected graph G, the following are equivalent:

1. G is a tree.
2. Every edge of G is a bridge.
3. G contains no cycles.
4. For all $a, b \in V(G)$, there is a unique path between a and b.
5. $|E(G)| = |V(G)| - 1$

★ For any graph G, equivalent are:

1. G is a forest.
2. Every edge of G is a bridge.
3. G contains no cycles.
4. NEW $\forall a, b \in V(C_G)$ where C_G is a component of G, there's a unique path from a to b.
5. NEW $|E(G)| = |V(G)| -$ (# of components)

(26) More Trees

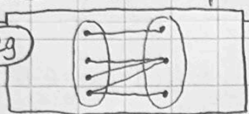
Tree Span Lemma

A Graph is connected IFF it has a spanning tree.

Bipartite Lemma

Forests are Bipartite.

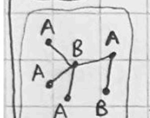
eg



★ **Proof Method: Labelling**

Eg: Prove any tree is bipartite.

Ans: Let T be a tree and fix $x \in V(T)$. Then for each $v \in V(T)$ there is a unique path between v and x.



Place v in A if the path has even length, and in B if it has odd length.

Exercise: Finish proof by showing A and B form a bipartition of T.

Note: prove things by splitting into components!

Bipartiteness Characterization Theorem

A graph is bipartite iff it has no odd-length cycles.

Jordan Curve Theorem

Every simple closed curve in the plane partitions $\mathbb{R}^2 \setminus C$ into two disjoint arcwise-connected sets.

↳ inner region: int(C)
↳ outer region: ext(C)

★ Every subgraph of a planar graph is planar. ★

Face-Shaking Lemma

$$\sum_{F \in F(G)} \deg(F) = 2|E(G)|$$

(27) Leaves + Planarity!

Leaf: a vertex v in a tree with deg(v) = 1.

Leaf Count Lemma

Let T be a tree with a vertex of degree d. Then T has ≥ d leaves.

Multigraph: has vertices V, and a multiset of edges where each edge is a pair of edges or a single edge (loop).

Plane: $\mathbb{R}^2$ (points have 2 real co-ordinates).

Curve (in $\mathbb{R}^2$): is the image of a continuous map $f: [0,1] \to \mathbb{R}^2$, where
↳ f(0) and f(1) are endpoints.

Closed Curve (in $\mathbb{R}^2$): a curve with $f(0) = f(1)$:

Simple Curve (in $\mathbb{R}^2$): a non-self-interesecting curve:

Simple Closed Curve: a closed curve that only intersects at its endpoints.

Planar Embedding: of a graph G consists of:

1. A point $x_v \in \mathbb{R}^2$ for every $v \in V(G)$ s.t. $x_v \neq x_{v'}$ if $v \neq v'$.
2. For each edge $uv \in E(G)$ with $u \neq v$, a simple curve (a "line", C_{uv}) with endpoints x_v, x_u.
3. For each loop at v, a simple closed curve containing x_v.
4. $\forall e \in E(G), \forall v \in V(G)$, $x_v \in C_e$ iff v is an endpoint of e.
5. For $e, e' \in E(G)$, C_e and $C_{e'}$ only intersect at shared endpoints.

Image: of G w/respect to a planar embedding is the set of points in $\mathbb{R}^2$ that are vertices or curves.

Planar Graph: a graph w/a planar (choice) embedding.

Plane Graph: a planar graph w/a fixed planar embedding.

Region: a region R of the plane (often a portion of $\mathbb{R}^2 \setminus$ Image(G)) is Arcwise-Connected if for all $a, b \in R$ there is a curve in R w/endpoints a, b.

(28) Spanning Trees + More Planarity

Span Cycle Lemma

Let G be a connected graph, let T be a spanning tree of G. Let $e \in E(G)$, $e \notin E(T)$. Then T+e has a unique cycle.

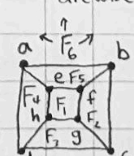
Further Lemma

Let $e' \in E(T+e)$ be an edge in the unique cycle of T+e. Then T+e−e' is a spanning tree of G.

Further Lemma

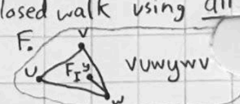
Let $f \in E(T)$. Let f' be an edge in the cut of G by a component of T−f. Then T−f+f' is a spanning tree of G.

Face of a Plane Graph G is a maximal arcwise-connected subset of $\mathbb{R}^2 \setminus$ Image(G).



Vertex (a) is incident with F_5.
Edge (ad) is incident with F_6.

Boundary Walk of a face F of a connected plane graph G is a closed walk using all edges incident with F.



deg(Face) = |boundary walk|

(let F(G) be the faces of G, a plane graph)

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28 cont'd Planarity

★ Facts to Use w/o Proof

1. Jordan Curve Theorem
2. Every subgraph of a planar graph is planar
3. If $uv \notin E(G)$ plane graph and u and v lie on the same face of G, then $G+uv$ is a plane graph.
4. If u and v are in the same component, $G+uv$ will get another face.
5. If $X \subseteq V(G)$ is a set of vertices all on the same face F, then you can add a vertex v in F adjacent to everything in X and get a plane graph.

29 More Planar Graphs

Euler's Formula

Let G be a planar graph. Then for any plane embedding of G:

$|V(G)|-|E(G)|+|F(G)|=2$

Lemma

Let G be a connected plane graph that is not a tree. Then for every face f of G, its boundary walk contains the the edges of a cycle

K_5 and $K_{3,3}$ are not planar.

Edge Subdivision a — b $\leftrightarrow$ a — c — b retains planarity in both directions.

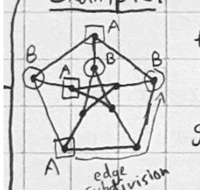
Kuratowski's Theorem A graph is not planar iff. K_5 and $K_{3,3}$ are subdivisions.

30 Planarity Proofs + Colouring

Demonstrating Planarity or Non-Planarity

1. Give a planar embedding
2. Find an edge subdivision of K_5 or $K_{3,3}$

Example:



found $K_{3,3}$ in our graph!

Deciding if a graph is colourable for $3 \leq k \leq n-1$ is in general an NP-complete problem.

K-Colouring: A k-colouring of a graph G is an assignment of a colour to each vertex of G where adjacent vertices have diff colour.

If C is a set of size k, then a k-colouring is a function $f: V(G) \to C$ s.t. $f(u) \neq f(v)$ $\forall uv \in E(G)$.

k-Colourable: if it has a k-colouring.

Extreme Cases

- Graph with no edges is 1-colourable
- (7.7.2) Bipartite graph $\leftrightarrow$ 2-colourable
- (7.7.3) K_n is n-colourable and isn't k-colourable for $k < n$.

Corallary 7.5.5

Every planar graph has a vertex of degree at most 5

31 Planar Colourings

6-Colour Theorem

Every Planar graph is 6-colourable.

5-Colour Theorem

Every Planar graph is 5-colourable.

4-Colour Theorem

Every Planar graph is 4-colourable.

32 Colouring, Matching

Contraction: G/e removes e and squishes its vertices together, preserving all edges (and planarity). G/e isn't necessarily a simple graph.

Matching: of a graph G is a set of edges in $E(G)$ where no two edges share a vertex.

Saturated: v in a matching M is saturated by M if v is incident with an edge in M.

Perfect Matching: saturates all $v \in V(G)$

33 More Matching

Alternating Path: w/respect to M is a path s.t. $e_i \in M$ iff $e_{i-1}, e_{i+1} \notin M$.

Augmenting Path: w/respect to M is an M-alternating path that starts/ends with non-M-edge, and first/last vertices not saturated.

Cover: of G is $S \subseteq V(G)$ s.t. every edge has a $v \in S$ endpoint.

König's Theorem

In a bipartite graph G, the size of a maximum matching is equal to the size of a minimum vertex cover.

LEMMA

In a bipartite graph, matching M having no augmenting path implies it's maximal

Maximum Matching: no matching exists w/more.

LEMMA

If S is a cover & M is a matching, then $|M|=|S| \to$ M is maximal, S is minimal

LEMMA

If S is a cover & M is a matching, $|M| \leq |S|$

33) Matching Cont'd

Algorithm Creation Method ★

XY-Construction: Bipartite graph (A B)

$X \subseteq A$ $\quad X_0$ = unsaturated vertices in A.

$Y \subseteq B$ $\quad$ X & Y are endpoints of alternating paths beginning in X_0.

- If $v \in Y$ is unsaturated, we have an augmented path!
- Otherwise:

$Y \cup (A \setminus X)$ is a vertex cover C.

Since all $v \in Y$ are saturated and all $v \in A \setminus X$ are saturated, $|C| = |M|$ and we found a max matching.

34) Applications

Hall's Theorem

Let G be bipartite with bipartitions $G = A \cup B$. Then G has a matching saturating A iff $\forall D \subseteq A$, we have $|N(D)| \geq |D|$ (where N(D) is all vertices adjacent to those in D).

Erdös-Reny Theorem

$\chi(G_n) \geq n$ w/ G_n triangle-free exists.

Vizing's Theorem

A graph G is always $(\Delta(G)+1)$-edge-colourable

Another Theorem

An individual colour in an edge colouring is always a matching.

35) More Matchings

Theorem

The k-regular bipartite graphs for $k > 0$ have perfect matchings.

Edge Colouring

$f: E(G) \to \{1, \dots, n\}$ such that if $e_1, e_2 \in E(G)$ share a vertex, $f(e_1) \neq f(e_2)$

$\chi(G)$ is the minimum # of colours to vertex-colour a graph.

$\chi(K_n) = n$

36) Final Review!

$\left(\!\binom{n}{k}\!\right)$ = # of ways to choose k objects of n types

$\hookrightarrow |(k_1, k_2, \dots, k_n)| = k$

$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{n-1}$

NBT

$$(1-x)^{-t} = \sum_{n=0}^{\infty} \binom{t+n-1}{t-1} x^n$$

Weight Function For Strings:

$w(x) = |x|$

Remember: • Block Decomposition • Recursive Expressions • Excluded substrings

Generating Series (non-negative formal power series)

$\sum_{k=0}^{\infty} c_k x^k$ w/ $c_k \geq 0$

$\sum_{s \in S} x^{w(s)} = \Phi_S^w(x)$

Weight function assigns each element a weight in $\mathbb{N}$.

String Lemma: $w(x) \neq 0$

Sum Lemma: must be disjoint

Rational Generating Series

- Write as a linear recurrence relation, or vice versa.
- Factor w/partial fraction decomp, then recurrence.

initial conditions $\leftrightarrow \frac{P(x)}{Q(x)} \leftrightarrow \frac{P(x)}{(1-cx)(1-dx)} \leftrightarrow C_k = Ac^k + Bd^k$

recurrence relations $\leftrightarrow \frac{P(x)}{Q(x)}$

$\frac{P(x)}{(1-cx)(1-dx)} \leftrightarrow$ exact formula for coefficients C_k.

Map of What We can Do!

Things to Know for Graph Theory

Component: maximally connected subgraph.

Handshaking Lemma: $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$

Degree of Vertex or Face: Face → walk about

Tree: Connected. No cycles $\overset{iff}{\longleftrightarrow}$ No bridges. Unique path between any v_1, v_2. $|V(G)| = |E(G)| + 1$.

Forest: Graph where every component is tree.

FaceShaking Lemma: $\sum_{f \in F(G)} d(f) = 2|E(G)|$ ← Planar-specific

Euler's Formula: $|V| - |E| + |F| = 2$ if connected.

Kuratowski's Theorem: Non-planar if $K_{3,3}$ or K_5 are edge subdivisions of the graph

Refresh on:

Cuts $\quad$ Colouring Proof Methods

Covers $\quad$ Planar Graph Theorems (from notes) a sugar glider

and a red panda

and some happy dogs on a couch

they’re happy because they don’t have to write a CO final 🪦