MATH 119 - W22
Calculus II for Engineers
Full Course Notes

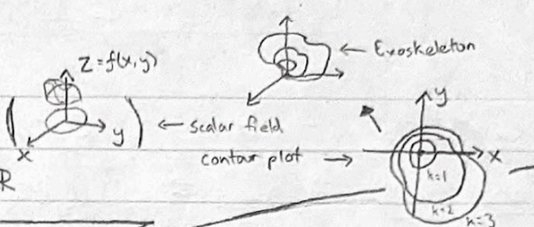
With Zack Cramer

Track down Zack Cramer's lecture videos; they are crazy good. My notes summarize that content.

[Josiah Plett](#)

MATH 119 Notes $\boxed{1}$

① Multivariable Functions $f: \mathbb{R}^n \to \mathbb{R}$



Graphing: use level curves (slices) for $f(x,y)=k,\ k \in \mathbb{R}$

2.1 Multivariate limits

Example: show $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ DNE...

Consider along $x=0$: $\lim_{(0,y)\to(0,0)} \frac{0}{0+y^2} = 0$

Consider along $y=0$: $\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+0} = 1$

Methods for Multivariate Limits

Proving DNE	Proving exists
find 2 paths that differ; start $y=0,\ x=0$, then $y=mx$, then $y=mx^2+cx$...	Convert to polar coor BECAUSE: we need to show, then: $(x,y)\to(0,0) \iff \rho \to 0^+$

Polar coor: $\rho = \sqrt{x^2+y^2}$, $\tan\phi = \frac{y}{x}$, $x=\rho\cos\phi$, $y=\rho\sin\phi$, so $x^2+y^2=\rho^2$:)

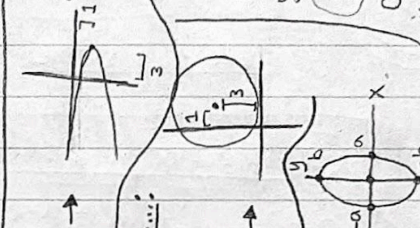
☆ only works when both x & y are going to 0 ☆

☆ make sure the ϕ expression ~~bounds~~ has ☆

CONIC SECTIONS

know to: • complete the square • factor

Parabolas: $x = 3-12(y+1)^2$

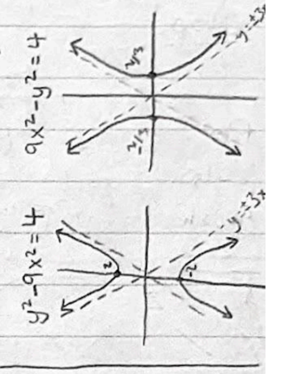


Circles & Ellipses...: $x^2+y^2-2x-6y+1=0 \Rightarrow (x-1)^2+(y-3)^2=9$ → CIRCLE

ELLIPSE: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (or $\frac{(x-c)^2}{a^2}+\frac{(y-d)^2}{b^2}=1$)

Hyperbolas: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; $y=0 \to x=\pm a$; asymptotes: $y=\pm\frac{b}{a}x$

Downwards vs sideways is a change of sign: $9x^2-y^2=4$, $y^2-9x^2=4$



Special case: hyperbolas that have equation $x^2-y^2=k$: asymptotes: $y=\pm x$ which are at 90°... 45° rotation → $xy=\frac{k}{2}$; $y=\frac{c}{x-a}+b$

? Continuity

Definition: A scalar field $z=f(x,y)$ is continuous at (a,b) in its domain if $\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)$

A function can't be continuous at a point that's not in its domain.

2.2 Partial Derivatives

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h} = f_x$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h} = f_y$$

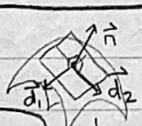
	FIRST time x	y
SECOND time x	$f_{xx}=\frac{\partial^2 f}{\partial x^2}$	$f_{yx}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$
y	$f_{xy}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$	$f_{yy}=\frac{\partial^2 f}{\partial y^2}$

Clairaut's Theorem

If f_x, f_y, and f_{xy} exist near (a,b) and f_{xy} is continuous at (a,b), then: $f_{yx}(a,b)=f_{xy}(a,b)$

③ Tangent Planes, Linear Approximation, & Differentials

Tangent Planes



Equation of our tangent plane:

$$n_1(x-x_0)+n_2(y-y_0)+n_3(z-z_0)=0$$

where $\vec{n}=\begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix}$ is the normal vector...

$$\vec{n}=\vec{d_1}\times\vec{d_2}=\begin{bmatrix}1\\0\\f_x(x_0,y_0)\end{bmatrix}\times\begin{bmatrix}0\\1\\f_y(x_0,y_0)\end{bmatrix}=\begin{bmatrix}-f_x(x_0,y_0)\\-f_y(x_0,y_0)\\1\end{bmatrix}$$

Thus, our real tangent-plane equation is:

$$z=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)+f(x_0,y_0)$$

Tangent Line equation: $y=f'(x_0)(x-x_0)+f(x_0)$

Linear Approximation

idea: $f(x_2,y_2)\approx L(x_2,y_2)$

Use the equation of the tangent plane that's nearby.

Differentials

idea: $\Delta f \approx df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

differential of f, of x, and y

Last step: divide by f, then: traingle inequality

$\left|\frac{\Delta x}{x}\right| \le 0.1$, $\left|\frac{\Delta f}{f}\right| \le ??$

④ Parametric Curves

(Parametrization) (split up vector components)

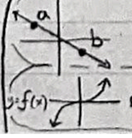
Instead of one equation that's dependent on 2 variables, say $x^2+y^2=1$, we do two equations dependent on one and the same variable:

② $x^2+y^2=1$

① $x(t)=\cos t$, $y(t)=\sin t$, $t\in(0,2\pi)$

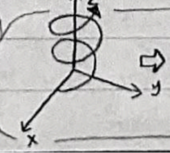
"Vector Function" ① $\vec{r}(t)=\langle \cos t, \sin t\rangle$ "Parametric Equation"

Examples:



$\Rightarrow \vec{r}(t)=\begin{bmatrix}x(t)\\y(t)\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}+t\begin{bmatrix}c-a\\d-b\end{bmatrix},\ t\in\mathbb{R}$

$y=f(x) \Rightarrow \vec{r}(t)=\begin{bmatrix}x(t)\\y(t)\end{bmatrix}=\begin{bmatrix}t\\f(t)\end{bmatrix},\ t\in\text{Domain}$



$\Rightarrow \vec{r}(t)=\begin{bmatrix}\cos t\\\sin t\\t\end{bmatrix},\ t\ge 0$

parametrized path speed: $\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$

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④ Calculus w/ Parametric Curves

position at time t:

VECTOR! $\vec{r}\,'(t_0) = \lim_{h \to 0} \frac{\vec{r}(t_0+h) - \vec{r}(t_0)}{h} = \begin{bmatrix} x'(t_0) \\ y'(t_0) \end{bmatrix}$

Tangent Line: $\vec{r}(t_0) + s \cdot \vec{r}\,'(t_0),\ s \in \mathbb{R}$

Slope of T.L.: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$\boxed{\text{eg}}$ if $\vec{r}(t) = \langle t^2, e^{2t}, 3 \rangle$, $t \in \mathbb{R}$, then $\vec{r}\,'(t) = \langle 2t, 2e^{2t}, 0 \rangle$

Position: $\vec{r}(t)$
Velocity: $\vec{r}\,'(t)$
Acceleration: $\vec{r}\,''(t)$

⑥ Chain Rule for Paths

$\frac{df}{dt} = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ where $\vec{r}(t) = \langle x(t), y(t) \rangle$

When x and y depend on t:

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

Lastly, replace your x's and y's with t's!

z
— differentiate
$\frac{\partial}{\partial}$ x y z
— differentiate
$\frac{d}{d}$ t t t
add add

⑦ More chain rule

Convert to Spherical: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

8.1 Directional Derivatives

– partial derivatives in a specific direction

USE UNIT VECTORS ($\|u\| = 1$)

$$D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{f(a+u_1 h,\ b+u_2 h) - f(a,b)}{h}$$

(where $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\|\vec{u}\| = 1$)

$$D_{\vec{u}} f(a,b) = f_x(a,b)\,u_1 + f_y(a,b)\,u_2$$

8.2 Gradients

UNIT VECTOR

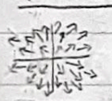
$$D_{\vec{u}} f(a,b) = \begin{bmatrix} f_x(a,b) \\ f_y(a,b) \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

So $f: \mathbb{R}^3 \to \mathbb{R}$, then $\nabla f = \langle f_x, f_y, f_z \rangle$

Gradient Vector (at (x,y)) is: $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

Equation of Tangent Line: $f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$ where $\vec{a} = \langle a,b \rangle$, $\vec{x} = \langle x,y \rangle$

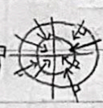
Gradient Plot:



① "Always in the direction of steepest ascent"

② "opposite the direction of the steepest descent"

③ "orthogonal to level curve containing that point"



$$D_{\vec{u}} f(a,b) = \|\nabla f(a,b)\| \cos\theta$$

between ∇f and $\vec{u}$

① Maximum of $D_{\vec{u}} f(a,b)$: $\|\nabla f(a,b)\|$

② Minimum of $D_{\vec{u}} f(a,b)$: $-\|\nabla f(a,b)\|$

⑨ Optimization

Critical point: $[f_x(a,b) = 0 \text{ or } = \text{DNE}]$ and $[f_y(a,b) = 0 \text{ or } = \text{DNE}]$

"Saddle point"

(multivariable)
Second derivative test:

— assume f_{xx}, f_{yy}, & f_{yx} exist & continuous at (a, b)

Discriminant: $\boxed{D(x,y) = f_{xx}f_{yy} - (f_{xy})^2}$ $\det\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ (Hessian matrix)

$D(a,b) > 0$: local extremum!
↳ $f_{xx}(a,b) > 0$ ➡ MIN
↳ $f_{xx}(a,b) < 0$ ➡ Max
OR f_{yy}

$D(a,b) < 0$: saddle point!

$D(a,b) = 0$: inconclusive :(

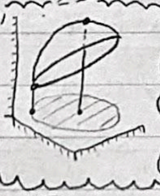
points / func	P_1	P_2
$f_{xx}=$		
$f_{yy}=$		
$f_{xy}=$		
D=		
conclusion		

⑩ Global Extrema

EVT: guaranteed global max/min

CIM: critical or end points (closed interval method)

★ Suppose D is closed & bouded region, and that $f: D \to \mathbb{R}$ is continuous ★



Process
① Find Critical Points
② Extreme values of the boundary
③ Plug in points!

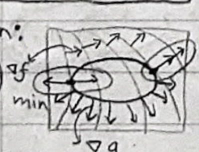
Lagrange Multipliers

Big Idea: — Global Max/Min — of f, subject to $g(x,y) = k$, occur when level curve of f is tangent to constraint curve.

At global max/min:

$$\nabla f = \lambda \nabla g$$

Lagrange Multiplier (unless $\nabla g = \vec{0}$!)



λ: sensitivity to constraint change

Process
① Find all (x, y) so $\nabla g(x,y) = \vec{0}$ & $g(x,y) = k$
② Find all (x, y) so $\nabla f = \lambda \nabla g$ & $g(x,y) = k$
③ Plug in everything from ① and ②!

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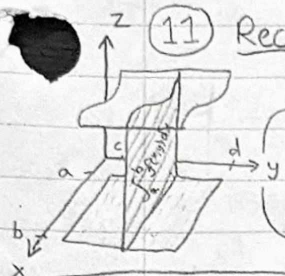
PARTS
$\int u\,dv = uv - \int v\,du$

IDENTITY
$\cos^2\phi = \frac{1}{2} + \frac{1}{2}\cos(2\phi)$
$\sin^2\phi = \frac{1}{2} - \frac{1}{2}\cos(2\phi)$

(11) Rectangles & Double Integrals (multivariable)

Concept: Volume $= \lim_{m,n\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i, y_j)\,\Delta x_i\,\Delta y_j$

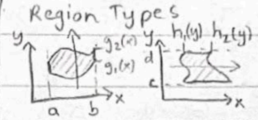


Volume $= \iint_R f(x,y)\,dA$

$= \int_c^d \int_a^b f(x,y)\,dx\,dy$

(12) General Regions & Double Integrals (multivariable)

Region Types



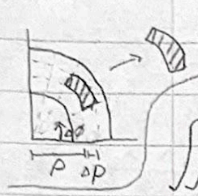
TYPE I   TYPE II

Type 1: $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

Type 2: $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

Difficult Integral?!
1. separate/swap integrals?
2. Convert from TYPE 1 $\Longleftrightarrow$ TYPE 2?

(13) Polar Co-or & Double Integrals



$dA = \rho\,\Delta\rho\,\Delta\phi$ $\rho = \sqrt{x^2+y^2}$

$\iint_R f(x,y)\,dA = \int_{\phi_1}^{\phi_2}\int_{\rho_1(\phi)}^{\rho_2(\phi)} f(\rho\cos\phi, \rho\sin\phi)\,\rho\,d\rho\,d\phi$

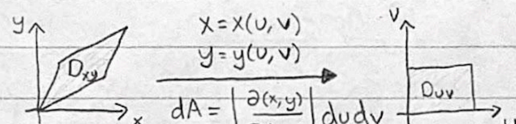
TRICK: Even functions $(f(x) = f(-x))$

eg $\int_{-\pi}^{\pi}(1-\cos 7\phi)\,d\phi = 2\int_0^{\pi}(1-\cos 7\phi)\,d\phi$

the whole idea!

Converting from Cartesian to Polar:
1. convert BOUNDS $\rightarrow$ graph the domain
2. convert FUNCTION $\rightarrow$ $x = \rho\cos\phi$ $y = \rho\sin\phi$
3. convert AREA FACTOR $\rightarrow$ $dA = \rho\,\Delta\rho\,\Delta\phi$

(14) Change of Variables



$x = x(u,v)$
$y = y(u,v)$

$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

$\det\begin{bmatrix} a & b \\ c & d\end{bmatrix} = ad - bc$

Jacobian

$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix}$

JUST notation

$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$

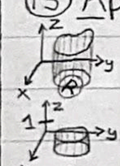
Change of Variables Strategy
1. Pick "good" transformation $(x,y) \to (u,v)$
2. Compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$
3. Determine bounds on u, v (sketch, or big brain pick)
4. Apply Change of Variables formula!

Formula

If $x = x(u,v)$, $y = y(u,v)$, and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ in D_{uv}, then:

$\iint_{D_{xy}} f(x,y)\,dA = \iint_{D_{uv}} f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

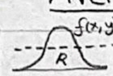
(15) Applications (of multiple integrals)



$\Rightarrow \iint_R f(x,y)\,dA = $ Volume

$\Rightarrow \iint_R 1\,dA = $ Area of R

Average Value



$f(x,y)$ --> average

$f_{avg} = \frac{1}{A}\iint_R f(x,y)\,dA$ (area) (ydA)

Total Amount

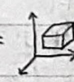
Let $f(x,y)$ represent density at (x,y) over a region R... then

$\iint_R f(x,y)\,dA = $ total amount

sometimes, $\rho = \frac{m}{A}$

triple applications

$E = $



Volume: $\iiint_E 1\,dV$

Average: $\frac{1}{V(E)}\iiint_E f(x,y,z)\,dV$

Amount: $\iiint_E f(x,y,z)\,dV$

(16) Triple Integrals

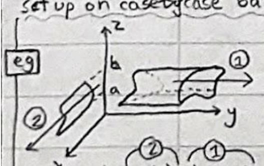
$\int_a^b\int_c^d\int_g^h f(x,y,z)\,\underbrace{dz\,dy\,dx}_{dV}$

TRIPLE!

(A) Applications!

set up on case by case basis

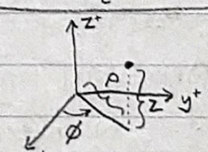
eg



$\int_a^b \int_{g_1(z)}^{g_2(z)} \int_{h_1(x,z)}^{h_2(x,z)} f(x,y,z)\,dy\,dx\,dz$

Graphing:
- outside $\to$ in
- reodering (dxdydz, dzdxdy)
- graph
- inside $\to$ out
  - after each, SMUSH!

(17) Cylindrical Coordinates



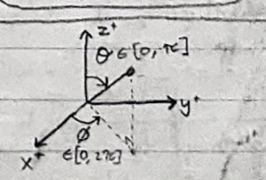
$(x,y,z) \to (\rho, \phi, z)$

- $x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$
- $\rho = \sqrt{x^2+y^2}$ $\tan\phi = \frac{y}{x}$ $z = z$

$\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \det\begin{bmatrix}\cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1\end{bmatrix} = \rho$

$dV = \rho\,dz\,d\rho\,d\phi$

(18) Spherical Coordinates



$(x,y,z) \to (r, \theta, \phi)$

$\theta \in [0,\pi]$ $\phi \in [0, 2\pi]$

- $x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$
  $z = r\cos\theta$

$\frac{\partial(x,y,r)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

3D Jacobian:

$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$

Trig Value:

$\int_0^{2\pi}\underbrace{\cos^2(\theta)}_{\text{or } \sin^2(\theta)}\,d\theta = 1$

MATH 119 Notes $\boxed{4}$ (second half of course $\geq$ (19) begins!)

(19) Newton's Method & Approximations

(1) Bisection Method
— always works (continuous) (steps)

consider $f(a)$ & $f(b)$. Then consider $f(\frac{a+b}{2})$. Continue analysis!

(2) Newton's Method
— fails: $\boxed{\text{Bad } x_0}$ $\boxed{\text{Not Converge}}$

Make initial guess x_0... (for $f(x)=0$)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(20) Polynomial Interpolation

Problem: Given data points, make a polynomial of degree n-1 curve through them! (n+1)

n^{th}-order polynomial through $(0, y_0), (1, y_1), \dots, (n, y_n)$ is:

$$y = y_0 + x\Delta y_0 + x(x-1)\frac{\Delta^2 y_0}{2!} + x(x-1)(x-2)\frac{\Delta^3 y_0}{3!} + \dots + x(x-1)\cdots(x-(n-1))\frac{\Delta^n y_0}{n!}$$

Finding $\Delta^n y_0$:

y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
y_1	Δy_1	$\Delta^2 y_1$	
y_2	Δy_2		
y_3			

"forward differences"

n^{th}-order polynomial through n+1 equidistant nodes:

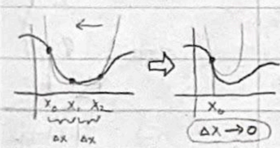
$x_0,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ \dots,\ x_n = x_0 + nh$

$$y = y_0 + \left(\frac{x-x_0}{h}\right)\Delta y_0 + \left(\frac{(x-x_0)(x-x_1)}{2!\,h^2}\right)\Delta^2 y_0 + \dots + \frac{(x-x_0)\cdots(x-x_{n-1})}{n!\,h^n}\Delta^n y_0$$

(21) Taylor Polynomials

Recall: Linear Approximation: $y = f(x_0) + f'(x_0)(x-x_0)$

Quadratic Approximation:



$$y = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2}$$

Higher order Approximation

$$y = P_{n,x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$$

"n^{th} order Taylor Polynomial of f centered at x_0."

(22) Shortcuts for Taylor Polynomials

Definition: "n^{th}-order Maclaurin Polynomial" is $P_{n,0}(x)$ (n^{th} order, centered at $x_0=0$)

Maclaurin's Theorem

If $P(x) = a_0 + a_1(x-x_0) + \dots + a_n(x-x_0)^n$ and $P^{(k)}(x_0) = f^{(k)}(x_0)$ for all $k = 0, 1, \dots, n$, then

$$P(x) = P_{n,x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$$

Shortcut!

$P(x) = n^{th}$ degree Maclaurin polynomial for $f(x)$ $\Rightarrow$ $P(kx^m) = m \cdot n^{th}$ degree Maclaurin polynomial for $f(kx^m)$

Tip: $x = x_0 \Rightarrow \boxed{P_{n,x_0}(x_0) = f(x_0)}$

Shortcut!

$P(x) = n^{th}$ degree Maclaurin polynomial for $f(x)$ at x_0 $\Rightarrow$

$P'(x) = P_{n-1,x_0}(x)$ for $f'(x)$

$\int P(x)\,dx = P_{n+1,x_0}(x)$ for $\int f(x)\,dx$

Be wary of $+C$!

Shortcut!

If $P(x)$ is the n^{th}-order Maclaurin polynomial for $f(x)$, then for all non-negative integers m, $x^m P(x)$ is the $(m+n)^{th}$-order Maclaurin polynomial for $x^m f(x)$.

(23) Taylor's Remainder Theorem

Q: How big is the error $|f(x) - P_{n,x_0}(x)|$?

$$f(x) - P_{0,x_0}(x) = f(x) - f(x_0) \overset{FTC}{=} \int_{x_0}^{x} f'(t)\,dt$$

← remainder! (error)

If $f(x) = P_{n,x_0}(x) + R_n(x)$, then:

$$R_n = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

(24) Taylor's Inequality

Bounding the error:

$$|R_n(x)| = \left|\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt\right| \leq \int_{x_0}^{x} \frac{|x-t|^n}{n!} \underbrace{|f^{(n+1)}(t)|}\,dt$$

$\to |f^{(n+1)}(t)| \leq K,\ t \in [x, x_0]$

$$|R_n(x)| \leq \frac{K(x-x_0)^{n+1}}{(n+1)!}$$

Triangle Inequality (for integrals)

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

only if $b>a$!

☆NOTE☆ round away from bound!

To find complicated max $|f(x)|$: eg: $\left|\frac{24 \cdot x \cdot (x^2-1)}{(1+x^2)^4}\right| = \frac{24 \cdot \overbrace{|x|}^{\leq \frac{1}{2}} \cdot \overbrace{|x^2-1|}^{\leq 1}}{\underbrace{|1+x^2|^4}_{\geq 1}} \leq 12$

Bounds: $x \in [-\frac{1}{2}, \frac{1}{2}]$

(25) Approximating Integrals

- Sandwich your integral between $P_{n,x_0}(x) \pm R_n(x)$ then compute!
- USE TAYLOR INEQUALITY!!!

When we add ∞ terms, we expect finite answer when:

(27)

(26) Taylor Series $\lim_{n\to\infty}$

Taylor series: $f(x) = P_{n,x_0}(x) + R_n(x) = \lim_{n\to\infty} P_{n,x_0}(x)$

infinitely many terms

If the remainder converges!

Partial sum

If $\{a_k\}_{k=0}^{\infty} = (a_0, a_1, a_2, \dots)$, then n^{th} partial sum is:

$$S_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + \dots + a_n$$

If it approaches (converges to) a finite number:

$$\sum_{n=0}^{\infty} a_n = S$$

If $\lim_{n\to\infty} S_n = \text{DNE}, \pm\infty$ then $\sum_{n=0}^{\infty} a_n$ diverges

Math 119 Notes $\boxed{5}$

$R = \lim_{k\to\infty} \left| \frac{C_k}{C_{k+1}} \right|$

(27) Geometric Series

$a + ar + ar^2 + ar^3 + \dots = \sum_{k=0}^{\infty} ar^k$

$S_n = \frac{a(1-r^{n+1})}{1-r}$

derive this yourself if you forget

So,
If $|r|<1$, $S_n \xrightarrow{n\to\infty} \boxed{\frac{a}{1-r}}$
If $|r| \geq 1$, S_n diverges

Test for Divergence

$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=0}^{\infty} a_n$ diverges

"n^{th} term test"

warning!
if $\lim_{n\to\infty} a_n = 0$, we don't know

p-series Test

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\leftarrow$ iff $\rightarrow$ $p > 1$

(28) Integral Test

IF $\int_{n\geq 1}^{\infty} f(x)\,dx$ converges,
THEN $\sum_{k=n\geq 1}^{\infty} f(k)$ converges;
ELSE $\sum_{k=n\geq 1}^{\infty} f(k)$ diverges

f is continuous, positive, decreasing

eg.
$\sum_{k=1}^{\infty} \frac{1}{k^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2}dx = 1 + \lim_{t\to\infty}\left[\frac{-1}{x}\right]_1^t = 1 + \left(\frac{-1}{\infty} + \frac{1}{1}\right) = 2$

Thus, since $\int_1^{\infty} \frac{1}{x^2}dx$ converges, so too does $\sum_{k=1}^{\infty} \frac{1}{k^2}$, and it's less-eq to 2!

VERY USEFUL

(29) Comparison Test

Direct Comparison Test

If Σa_n, Σb_n are series of positive terms and $a_n \leq b_n$ for all n:
1. Σa_n diverges $\Rightarrow \Sigma b_n$ diverges
2. Σb_n converges $\Rightarrow \Sigma a_n$ converges

Limit Comparison Test

If Σa_n, Σb_n are series of positive terms and if $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, then either both converge or both diverge!

(30) Alternating Series Test (AST)

Suppose we have $\sum_{k=0}^{\infty} (-1)^k b_k$,
IF (i) $\{b_k\}$ is a decreasing sequence
(ii) $\lim_{k\to\infty} b_k = 0$

THEN (i) $\sum_{k=0}^{\infty} (-1)^k b_k$ converges

*** ASET *** $\rightarrow$ (ii) $\underbrace{|S - S_n|}_{\text{error}} \leq b_{n+1}$; $S_n = n^{th}$ partial sum

Absolute vs Conditional Convergence

Σa_n converges "absolutely" iff $\Sigma |a_n|$ converges

Also: "Σa_n converges absolutely" $\Rightarrow \Sigma a_n$ converges

Σa_n converges "conditionally" iff $\Sigma |a_n|$ diverges, Σa_n converges

Also: "Σa_n converges conditionally" $\Rightarrow$ for any $\alpha \in \mathbb{R}$, there is a rearrangement of Σa_n that converges to α!

RRT

"Σa_n converges absolutely" $\Rightarrow$ every rearrangement converges to the same sum

(31) Ratio Test

Suppose $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists (or $=\infty$)
1. if $L<1$, Σa_n converges absolutely
2. if $L>1$, Σa_n diverges
3. if $L=1$, totally inconclusive

Root Test

eats factorials for breakfast

$f(n)^{g(n)}$

Suppose $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists (or $=\infty$)
1. if $L<1$, Σa_n converges absolutely
2. if $L>1$, Σa_n diverges
3. if $L=1$, totally inconclusive

USEFUL LIMITS

for $c>0$, $\lim_{n\to\infty} \sqrt[n]{c} = 1$
for $p>0$, $\lim_{n\to\infty} \sqrt[n]{n^p} = 1$
$\lim_{n\to\infty} \sqrt[n]{\ln(n)} = 1$
$\lim_{n\to\infty} \sqrt[n]{n!} = \infty$

(32) Power Series

- A power series centered at x_0 is of the form $\sum_{n=0}^{\infty} C_n (x-x_0)^n$

Radius & Interval of Convergence

x_0 is the "center" of your power series.. the "radius" is measured from x_0. "interval of convergence," then is $x \in (x_0 - r, x_0 + r)$ OR [ ] (further tests)

(33) Manipulating Power Series

If P power series $(\Sigma C_n (x-x_0)^n)$ has radius of convergence R:
- differentiate term-by-term
- integrate term-by-term
- multiply by non-zero constant

and the R won't change!

But the interval might! endpoints

Binomial Formula

$m \in \mathbb{N}$

$(1+x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n = \sum_{n=0}^{m} \frac{m!}{(m-n)!\,n!} x^n$

Given $m \in \mathbb{R}$, $m \notin \mathbb{N}$, the binomial series for $(1+x)^m$ is given by:

$$(1+x)^m = \sum_{n=0}^{\infty} \underbrace{\frac{m(m-1)(m-2)\dots(m-n+1)}{n!}}_{\binom{m}{n}} x^n$$

WITH $R=1$

$P_{n,c} = \sum^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \leftarrow$ for $|x-c|<R$, $R>0$

EXPANSION:

Math 119 Notes 6

(34) Big-O Notation

Definition: f is of the order g as $x \to x_0$ if $\exists c \in \mathbb{R}$ such that

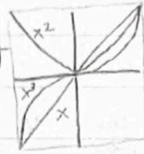
$$|f(x)| \le c|g(x)| \iff f(x) = O(g(x)) \text{ as } x \to x_0$$

↳ for all x near x_0, but not necessarily at x_0

Algebra: ★ as $x \to 0$ ★

1. $k\,O(x^n) = O(x^n)$, for any constant k
2. $O(x^m) + O(x^n) = O(x^q)$, $q = \min(m, n)$
3. $O(x^m) \cdot O(x^n) = O(x^{m+n})$
4. $[O(x^n)]^m = O(x^{mn})$
5. $\dfrac{O(x^m)}{x^n} = O(x^{m-n})$

eg



$|x^3| \le 1|x^2|$ on $[-1, 1]$ so:

$x^3 = O(x^2)$ as $x \to 0$

$x^3 = O(x)$ as $x \to 0$

$kx^3 = O(1)$ as $x \to 0$

for all $k \in \mathbb{R}$!

From Taylor's Inequality, we have

$$R_n(x) = O((x-x_0)^{n+1}) \text{ as } x \to x_0$$

$$f(x) = P_{n,x_0}(x) + O((x-x_0)^{n+1})$$

Evaluating Limits (with Taylor Series)

1. Write out the Maclaurin Series for the functions in your limit
2. Use Big-O, and then get some cancellation
3. Evaluate the simpler limit!

ideal study strategy