MATH 117 - F21
Calculus I for Engineers
Full Course Notes

With Prof Mohammad Kohandel

I took these notes before I got good at during-lecture notes. Nevertheless they should be all you need.

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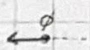
MATH 117

FALL 2021 $\boxed{\#1}$

go through the material in this course
by myself (do ~ practice)



(1.1) Defining Functions

"A dependance of one quantity on another" (-∞, 0].
Functions MUST pass the vertical-line test. ☆ $\pm\sqrt{x}$ or $\sqrt{x}$? ☆

(1.2) Composition of Functions

D_g = domain of g
D_f = domain of f

☆ Use Interval Notation

IMPORTANT FACT: (about domains)

$(f\circ g)(x) = f(g(x)) \longrightarrow$

$$D_{f\circ g} = \begin{cases} x \in D_g \\ g(x) \in D_f \end{cases}$$

NEW RULES:

$(f\pm g)(x) = f(x) \pm g(x)$
$(fg)(x) = f(x)g(x)$
$(f/g)(x) = f(x)/g(x) \setminus \{x \mid g(x)=0\}$
$(f\circ g)(x) = f(g(x))$

excluding

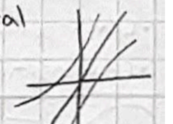
for each, we need ~~only~~ the union of their domains: $D_f \cap D_g$

(1.3) Inverse Functions

$x = f(x)$
$f(x) = x$

$f^{-1}(x)$ is inverse

horizontal line test!



Only inversible if the function is 1–1 (if it never takes the same value twice).

Fun fact: $f(x)=y \equiv f^{-1}(y)=x$ ‖ When done, swap variables back, unless they have physical meaning

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(1.4) Symmetry of Functions

Even: symmetrical across y-axis, $f(-x) = f(x)$

Odd: symmetrical about origin, $f(-x) = -f(x)$

Any Function whose domain is symmetric about $x=0$

$$\overbrace{f(x)} = \underbrace{\tfrac{1}{2}(f(x)+f(-x))}_{\text{Even}} + \underbrace{\tfrac{1}{2}(f(x)-f(-x))}_{\text{Odd}}$$

Even + odd: neither
Even x odd: odd
Even / odd: odd
odd x odd: even
Odd (Even): even
Even (Odd): even
Any (Even): even
Odd (odd): odd
even (even): even

Func	D	R
$\sin^{-1}(x)$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}(x)$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$\cot^{-1}(x)$	"	$(0, \pi)$
$\sec^{-1}(x)$	" not $(-1,1)$	$[0, \pi]$ not $\pi/2$
$\csc^{-1}(x)$	"	$[-\frac{\pi}{2}, \frac{\pi}{2}]$ not 0

$$e^x = \underbrace{\tfrac{1}{2}(e^x + e^{-x})}_{\cosh x} + \underbrace{\tfrac{1}{2}(e^x - e^{-x})}_{\sinh x} = \cosh x + \sinh x$$

ln laws
$\boxed{eg}$

$\ln(e^5) + \ln(e^6) = \ln(e^{11}) = 11$

(1.5) Piecewise Functions

$\boxed{eg}$ $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

To solve piecewise inequalities (eg. $|x+3| \leq |2x+1|$),
Break them up into many cases.

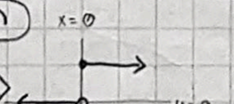
(5.3) Floor/ceiling Functions

$\lfloor 4.2 \rfloor = 4$ $\lfloor -2.3 \rfloor = -3$

$\lceil -2.3 \rceil = -2$ $\lceil x \rceil = -\lfloor x \rfloor$

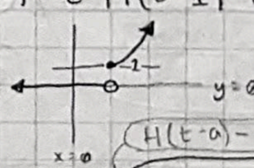
(5.6) Heaviside Function

$$H(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0 \end{cases} \Rightarrow$$



Used to formulate any piecewise function!

$\boxed{eg}$ $f(t) = t^2 H(t-1)$ is



Writing in Heaviside form

$$\boxed{eg}\ f(t) = \begin{cases} -t & \text{for } t < 0 \\ t^2 & \text{for } 0 \leq t < 2 \\ 4 & \text{for } t \geq 2 \end{cases} \Rightarrow$$

$f(t) = -t$
$f(t) = -t + (t + t^2)H(t)$
$f(t) = -t + (t+t^2)H(t) + (4-t^2)H(t-2)$

$H(t-a) - H(t-b)$

$$\underbrace{(f(t))\left[H(t-a) - H(t-b)\right]}_{\text{On/off switch for } a \leq t < b}$$

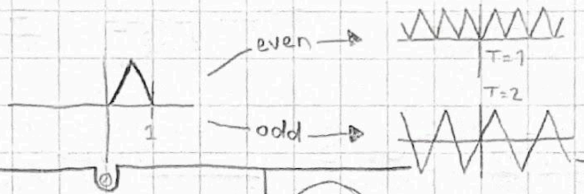
(1.6) (6) Periodicity

A function $f(t)$ is periodic if there is a number T such that $f(t+nT)=f(t)$ for every integer n. T is the period.

T = Period

f = Frequency = $\frac{1}{T}$

ω = Angular Frequency = $\frac{2\pi}{T}$



HEAVISIDE:

$$H(x) := \begin{cases} 0, & x<0 \\ 1, & x\geq 0 \end{cases}$$

ACHIEVE

1.3: Classes of Functions

- Polynomials $P(x)$
- Rational functions $\frac{P(x)}{Q(x)} = R(x)$
- Algebraic functions $\sqrt{} + R(x) = A(x)$
- Exponential functions $b^x = E(x)$
- Trigonometric functions $\frac{\sin(x)}{\cos(x)} = T(x)$
- Non-algebraic = Transcendental

- Linear combination of f and g $= c_1 f(x) + c_2 g(x)$
- Composition: $(f \circ g)(x) = f(g(x))$

1.4: Trigonometric Functions

☆ always in RADIANS ☆

simple Trig

(1.8) (8) Trigonometric Functions

arc length $s = r\theta$

MUST MEMORIZE:

$\boxed{\cos^2\theta + \sin^2\theta = 1}$

÷ $\cos^2\theta$

$\boxed{\sec^2\theta = 1 + \tan^2\theta}$

$\boxed{\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta}$

$\boxed{\sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta}$

$\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$

$\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$

$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}$

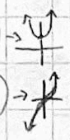
$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$

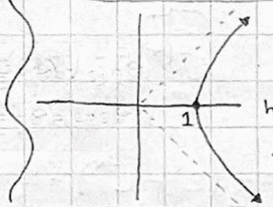
$\boxed{\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)}$

$\boxed{\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)}$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$





The "unit" hyperbola

$\boxed{\cosh^2 x - \sinh^2 x = 1}$

$\boxed{\tanh x = \frac{\sinh x}{\cosh x}}$

(1.7) (7) Rational Functions & Partial Fraction Decomposition

"Proper": numerator degree < denominator degree.

AKA PFD

DECOMPOSITION METHOD:

1. Factor denominator as far as possible.
2. Predict the form of the PFD:
   - A. For LINEAR denom factor: term is $\frac{A}{c_1x + c_0}$
   - B. For QUADRATIC denom factor: term is $\frac{Ax+B}{c_2x^2 + c_1x + c_0}$
   - C. For REPEATED factors (n times): n terms with 1 through n exponents
3. Make common denominator again.
4. Cancel denoms, expand, & match coefficients.
5. Solve for the constants! (congrats)

Long division with Polynomials

$$x+4 \overset{①}{\overline{\big)\, x^2 + 7x - 9}}$$
$-②$

① Divide: $\frac{x^2}{x}$

② Multiply: $x(x+4)$

③ Subtract.

④ Repeat!

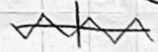
Synthetic division with Polynomials

$$\frac{x^2+7x-9}{x+4} \Rightarrow \begin{array}{r|rrr} -4 & 1 & 7 & -9 \\ & \downarrow & -4 & -12 \\ \hline & 1 & 3 & -21 \end{array}$$

⇓

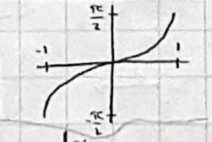
answer → $x + 3 - \frac{21}{x+4}$

$y = \sin^{-1}(\sin(x)) =$

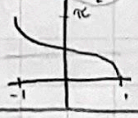


(1.9) Inverse Trigonometric Functions

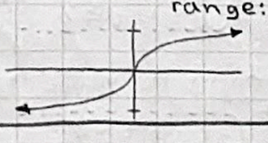
$\sin^{-1} x$ domain: $[-1, 1]$ range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$\cos^{-1} x$ domain: $[-1, 1]$ range: $[0, \pi]$



$\tan^{-1} x$ domain: $(-\infty, \infty)$ range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

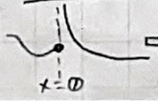


$\sinh^{-1}(x) = \ln(x + \sqrt{x+1})$

$\boxed{\sin(\cos^{-1}(x)) = \sqrt{1-x^2} = \cos(\sin^{-1}(x))}$

Infinite Discontinuity: any vertical asymptote.

Left continuous at $x=c$ if $\lim_{x\to c^-} f(x) = f(c)$



⇒ infinite discontinuity at x=0

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(1.10) Combining Sine & Cosine

When we have $f(t) = \overset{\text{amplitude}}{B}\sin(\overset{\text{angular frequency}}{\omega} t)$,

output is often $a\cos(\omega t) + b\sin(\omega t)$. sine wave!

We try to rewrite as $A\sin(\omega t + \overset{}{\alpha})$... phase shift

STRATEGY:

$\boxed{1}$ Do we want $A\underline{\sin}(\omega t+\alpha)$ or $A\underline{\cos}(\omega t+\alpha)$?

$\boxed{2}$ Use the sum-of-angles identities...

$\boxed{3}$ Equate coefficients of $\sin(\omega t)$ & $\cos(\omega t)$

as a and b

$\boxed{4}$ Square and add together: $A = \sqrt{a^2+b^2}$

$\boxed{5}$ Determine α via inverse trig functions:

use sign to determine quadrant

$\cos^{-1}$	$\sin^{-1}$ $\tan^{-1}$
$\tan^{-1}$ and shift by π	$\sin^{-1}$ $\tan^{-1}$

Which functions to use, per quadrant

Having understood this process, it might be nice to know the shortcut way:

$$a\sin(\omega t) + b\cos(\omega t) = \sqrt{a^2+b^2}\,\sin\left(\omega t + \tan^{-1}\left(\frac{b}{a}\right)\right)$$

However, use the signs of $\sin(\alpha)$ and $\cos(\alpha)$ to properly define $\tan^{-1}\left(\frac{b}{a}\right)$.

LIM ⇩ Multiply by Conjugate

☆ $\lim_{h\to 0} \frac{1-\cos h}{h} = 0$

☆ $\lim_{h\to 0} \frac{e^h-1}{h} = 1$

(2.1) Limits of Sequences → simply an ordered list.

Facts about sequences

- Can be finite or infinite
- Denoting a sequence: $\{a_n\}_{n=0}^{\infty}$ or just $\{a_n\}$
- Some sequences can be defined by formulas.
- The formula for a_n is called "the general term."
- Some seq. are defined recursively :/.
- Convergent: defined limit.
- Divergent: no defined limit.

$\boxed{\text{eg.}}$ general term $a_n = 1 - \frac{1}{n}$.
We can say a_n tends towards 1:

$\boxed{a_n \to 1 \text{ as } n \to \infty}$

main → $\boxed{\lim_{n\to\infty} a_n = 1}$

Formal definition:

a_n converges to L if $\forall \underset{}{\varepsilon} \in \mathbb{R}, \exists N \in \mathbb{Z}\ [n > N \Rightarrow |a_n - L| < \varepsilon]$

positive, though!

2.1 but better Calculating Limits

☆ $\lim_{n\to 0} \frac{\sin n}{n} = 1$

☆ $\lim_{n\to c} n = c$

☆ $\lim_{n\to 0} \frac{1-\cos n}{n} = 0$

☆ $\lim_{n\to\infty} c = c$

☆ $\lim_{n\to\infty} n = \infty$

☆ $\lim_{n\to\infty} \frac{1}{n} = 0$

☆ limit of a product is equal to the product of the limits

↓

☆ same with quotients

☆ $\lim_{n\to 0} \frac{e^n-1}{n} = 1$

☆ $\lim_{n\to\infty} \frac{1}{n^p} = 0$ (for all $p > 0$)

☆ if $|r| < 1$, $\lim_{n\to\infty} r^n = 0$

☆ $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$

if f is continuous and $\lim_{n\to\infty} a_n$ exists.

☆ if $a_n \to L$ and $b_n \to L$ as $n\to\infty$, and if $a_n \le b_n \le c_n$, then $\lim_{n\to\infty} b_n = L$.

LIMIT LAWS

see ↑ for more :)

$$\lim_{\infty} \frac{a}{b} = \frac{\lim_{\infty} a}{\lim_{\infty} b}$$

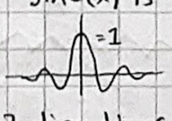
(12) Limits of Functions in $\mathbb{R}$

→ Course notes!

The statement "$f(x) \to L$ as $x \to a$" means that for any $\varepsilon > 0$, there exists δ such that

$$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Fact: sinc(x) is



☆ f is continuous at a value a if and only if $f(x) \to f(a)$ as $x \to a$. ☆

WAYS OF EVALUATING LIMITS

INFINITE?!

① Plug in. :)
② Not work? Consider approaching from 2 directions.
③ Not work? Algebra to make denominator safe.
④ Bounded? $\boxed{\text{eg}}$ $\lim_{x\to 0} x\cos(\frac{1}{x}) = 0$ Use boundary for $\boxed{\text{squeeze theorem!}}$
⑤ Piecewise? Check both side limits.
⑥ Irrationally appoaching ∞? Multiply by conjugate and divide by power of x.
⑦ Approaching NEGATIVE ∞? Consider negativify during algebra.

$\boxed{\text{form of } \infty - \infty}$

(2.9) Limit Definition $\varepsilon - \delta$

→ Achieve Textbook

$\boxed{\text{STEP 1}}$ Relate the Gap (to $|x-a|$)

$\boxed{\text{STEP 2}}$ Choose δ (in terms of ε)

☆ if need be, APPROACH PIECEWISELY ☆

⇩

Assume $|x-a| < 1$, and $\boxed{\text{create}}$ inequalities based off that... then tie together at end with $\min(1, \boxed{\downarrow})$

expression with ε for δ

⑬ Continuity

$f(x)$ is continuous on an interval if for any x and y in that interval, and for any positive number ε, there exists a number δ such that

$$|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$$

$f(x)$ is continuous at $x = a$ if

$\lim_{x\to a} f(x)$ exists

and

$\lim_{x\to a} f(x) = f(a)$

Continuity Theorems

If $f(x)$ & $g(x)$ are continuous on the interval I.

- $(f \circ g)(x)$ is continuous on I.
- $(f \pm g)(x)$ are continuous on I.
- $(fg)(x)$ is continuous on I.
- $1/f(x)$ is continuous when $f(x) \neq 0$

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

Points to Remember:

$\boxed{1}$ Essentially all functions we see are continuous ON THEIR DOMAINS.

$\boxed{2}$ Continuity will pretty much only be in question when we are dealing with piecewise funcs.

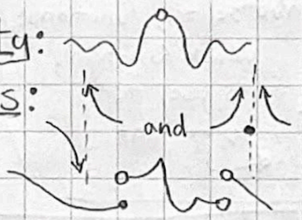
→ f is continuous at a if and only if $f(x) \to a$ as $x \to a$.

Types of Discontinuity:

Removable Discontinuity: (can be fixed by defining a single value)

Infinite Discontinuities: (when at least one side $= \infty$)

Jump Discontinuities: (when $\lim_{x\to a^-} \neq \lim_{x\to a^+}$, but not ∞)



IVT Intermediate Value Theorem

If c is between $f(a)$ and $f(b)$, then there exists x between a and b where $f(x) = c$.

Application: Root Finding: $\boxed{\text{EG}}$ $e^x - 2 = \cos(x)$ let $f(x) = e^x - 2 - \cos(x)$. Find roots by testing values for x until you have $a \leq f(x) \leq b$ as precise as you want.

Application: Curve sketching: sign analysis

EVT Extreme Value Theorem

If f is continuous on $[a, b]$, then f attains a defined max and min value for $a \leq x \leq b$.

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⑭ The Derivative

Equation of Tangent Line:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Common Notation:

Lagrange's Notation: → $f'(x_0)$

Leibniz's Notation: → $\frac{dy}{dx}(x_0)$ → $\frac{d}{dx}(f(x_0))$

evaluate: $\frac{dy}{dx}\Big|_{x=a}$

Newton's Notation: → $\dot{f}(x_0)$ → sometimes preferred in physics

Euler's Notation: → Df

second	fifth
$f''(x)$	$f^{(5)}(x)$
$\frac{d^2y}{dx^2}$	$\frac{d^5y}{dx^5}$
$\ddot{f}(x)$	N/A

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Difference Quotient Approximation: $f'(a) \approx \frac{f(a+h) - f(a)}{h}$

Tangent Line at a (of $f(x)$): $y = f'(a)(x-a) + f(a)$

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

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(3.1) $\boxed{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}$ ← derivative!

NOT DIFFERENTIABLE:
- Different-sided limits
- Incalculable extreme
- Infinity, (not in domain)

(3.2) Leibniz Notation: $\left.\boxed{\frac{df}{dx}}\right|_{x=6} = 3(6)^2$ if $f(x) = x^3$

☆Differentiability implies continuity☆

☆ If $f'(a)$ exists, f is <u>locally linear</u> at a☆

$\frac{d}{dx} c = 0$

$\frac{d}{dx} x^n = n x^{n-1}$

$\frac{d}{dx} f+g = f'+g'$

$\frac{d}{dx} e^x = e^x$

$\frac{d}{dx} fg = f'g + fg'$

$\frac{d}{dx} \frac{f}{g} = \frac{f'g - fg'}{g^2}$

⑪ $\sin(x) = \cos(x)$

$\cos(x) \to -\sin(x)$

$\tan(x) \to \sec^2(x)$

$\cot(x) \to -\csc^2(x)$

$\sec(x) \to \sec(x)\tan(x)$

$\csc(x) \to -\csc(x)\cot(x)$

$f(g(x)) \to f'(g(x))\, g'(x)$

$\sin^{-1}(x) \to \frac{1}{\sqrt{1-x^2}}$

$\cos^{-1}(x) \to \frac{-1}{\sqrt{1-x^2}}$

$\tan^{-1}(x) \to \frac{1}{x^2+1}$ ($\cot^{-1}(x)$ negative)

$\sec^{-1}(x) \to \frac{1}{|x|\sqrt{x^2-1}}$ ($\csc^{-1}(x)$ negative)

$\frac{d}{dx} \ln x = \frac{1}{x}$ (for $x > 0$)

$\sinh(x) \to \cosh(x)$

$\cosh(x) \to \sinh(x)$

$(f^{-1})'(x) \to \frac{1}{f'((f^{-1})(x))}$

(3.4) $\boxed{\frac{\Delta y}{\Delta x} = \frac{f(x_1)-f(x_0)}{x_1 - x_0}}$ ← average rate of change!

(3.5) <u>Higher Derivatives</u>

$\frac{df}{dx} \to \frac{d^2f}{dx^2} \to \frac{d^3f}{dx^3}$

(3.8) <u>Implicit Differentiation</u>

$\boxed{\text{Ex}}$ $y^4 + xy = x^3 - \sin(y^2)x \longrightarrow \frac{d}{dx}4y + y + \frac{d}{dx}x = 3x^2 - (\cos(y^2)\frac{d}{dx}2yx + \sin(y^2))$

$\boxed{\frac{dy}{dx}(4y + x + 2xy\cos(y^2)) = 3x^2 - y - \sin(y^2)}$

<u>Higher order derivatives</u> (Implicit)

① Differentiate Implicitly; solve for y'.
② Differentiate implicitly again; sub in y'.
③ DON'T FORGET to "un-sub" in y! (usually you can)

Generalized Power Rule

$\frac{d}{dx} f(x)^{g(x)} = f(x)^{g(x)} (\ln(f(x))\, g(x))'$

(3.9) *Exponetial <u>Funcs</u>!* >>>

$\boxed{f^g \to f^g\left(\frac{f'g}{f} + \ln(f)\,g'\right)}$

(3.5) Mobius <u>Logarithmic Differentiation</u>:

take ln of both sides, then use ~~power~~ log rules to simplify, then do <u>implicit differentiation</u>.

(3.6) <u>Theorems</u>

- <u>Differentiability implies Continuity</u>.
- $\boxed{\text{MVT}}$ ||| $\exists$ Ⓒ $\in (a,b)$, $\boxed{f'(c)} = \frac{f(b)-f(a)}{b-a}$ if f is continuous on $[a, b]$ and differentiable on (a, b).
- <u>L'Hôpital's Rule</u> CONDITIONS:
  ① Form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$
  ② f and g are differentiable on open interval I. (except perhaps at a)
  ③ True as long as $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ actually exists.

  $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$
- $\boxed{0 \cdot \infty}$ <u>Indeterminate products</u>: Write in L'Hôpital's form!
- $\boxed{\infty - \infty}$ <u>Indeterminate differences</u>: ↳ also, make common denominators.

(3.7) <u>Related Rates</u> (x and y)

(<u>Course Notes</u> Ch ⑱

Suppose we have two variables related by $y = f(x)$, then the rates of change are also related, by: $\boxed{\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt}}$

WORD PROBLEMS?? (☆or other!)
Name variables and constants, relate them, & differentiate w/ respect to time☆

(3.8) <u>Linear Approximation</u>

- Let the "Linearization" (tangent line to) of $f(x)$ at $x = a$ be La: $\boxed{L_a^f(x) = f(a) + f'(a)(x-a)}$

(3.9) Mobius <u>Differentials</u>

(<u>Course Notes</u> Ch ⑲)

$\left\{\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}\right\} \Rightarrow$ MEANING

dx be "independent" and $dy = f'(x)\,dx$ ...
dy and dx are only defined in relation to each other.

$\boxed{f(x_0 + \Delta x) \approx f(x_0) + f'(x)\Delta x \approx \Delta f}$

<u>Critical Points</u>: $\boxed{f'(x) = 0}$
↳ Testing

<u>Monotonic</u>: either increasing or decreasing.

3.10 Mobius Local/Global Extrema

- **Absolute/Global Maximum:** $f(c)$ is max if $f(c) \geq f(x)$ for all $x \in$ Interval.
- **Absolute/Global Minimum:** $f(d)$ is min if $f(d) \leq f(x)$ for all $x \in$ Interval.
- **Local Max & Min:** As above ↑, but only for x near c/d.

Critical Point:
$c = x$ is a critical point if:
$f'(c) = 0$ or $f'(c)$ DNE

Eg $f(x) = x$, $x \in (1, 3)$ has NO EXTREMA!

EVT: A closed-bounded function always has a Global max and min.

Fermat's Theorem:
Local Extrema can ONLY occur at critical points OR endpoints of interval.

Get Extrema! (closed interval method)
① Get the derivative: $f'(x)$.
② Determine all critical points in interval.
③ Evaluate all critical points: $f(c)$.
④ Largest ↑ = Global Max, smallest = min.

(on a given interval)
- **Increasing:** if $f'(x) > 0$
- **Decreasing:** if $f'(x) < 0$
- **Monotonic:** if (on interval) $f(x)$ is strictly increasing or decreasing

First derivative test:
- local max: $f'(x)$ goes from ⊕ to ⊖.
- local min: $f'(x)$ goes from ⊖ to ⊕.
- Neither: $f'(x)$ doesn't change sign.

3.11 Mobius Second Derivative

- **Concavity:** "Concave Up" = $(f''(x) > 0)$ & "Concave Down" = $(f''(x) < 0)$
- **Points of Inflection:** c is, if $f''(c) = 0$ or $f''(c)$ is undefined, AND $f''(x)$ switches sign when crossing c.

☆ points that don't exist can't be extrema! ☆

Second derivative test:
- local max: $f'(c) = 0$ and $f''(c) < 0$.
- local min: $f'(c) = 0$ and $f''(c) > 0$.

4.1 Area Under Curve! (& summations...)

$$\sum_{i=1}^{n} 1 = n$$

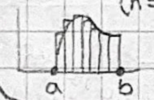
$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

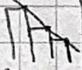
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} r^i = \frac{r - r^{n+1}}{1-r} \quad (r \neq 1)$$

Rieman Sums
(n=7 here)



Width: $\Delta x = \frac{b-a}{n}$

Right RS: 

Left RS: 

Middle RS: 

4.2 Definite Integrals

$$\Delta x = \frac{b-a}{n}$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} (\tilde{x}_i) \cdot \Delta x\right]$$

where: $x_i = f(a + i\Delta x)$
(and each $\tilde{x}_i \in [x_{i-1}, x_i]$.)
that is, $\tilde{x}_i = f(a + \frac{ib}{n})$

"dummy" variable x: only used to connect the
- bounds,
- integrand, &
- differential.

The integrand is $f(x) = 1$:

$$\int_a^b 1\,dx = \int_a^b dx = b - a$$

If $f(x)$ is odd and $a = -b$:

$$-\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_b^a f(x)\,dx = 0$$

$$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_a^c f(x)\,dx$$

4.3 (FTC) Fundamental Theorem of Calculus

- If $A'(x) = f(x)$, then $A(x)$ is the "antiderivative"
- Let $F(x)$ be general antiderivative of $f(x)$: $F(x) = A(x) + C$

FTC 1 $A(x) = \int_a^x f(t)\,dt \Rightarrow A'(x) = f(x)$
(if $f(x)$ is continuous on $[a, b]$)

FTC 2 $\int_a^b f(x)\,dx = F(b) - F(a)$ $(F'(x) = f(x))$
(if $f(x)$ is continuous on $[a, b]$)

For $x \in [a, b]$

$f(x) \geq g(x)$ for $x \in [a, b]$, then
$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$ and vice versa

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)(x')$$

$$\int_{h(x)}^{g(x)} f(t)\,dt = \int_0^{g(x)} f(t)\,dt - \int_0^{h(x)} f(t)\,dt$$

if $G(t) = \int_a^{g(t)} f(x)\,dx$ then → $G'(t) = f(g(x))\,g'(x)$

$G(t) = A(g(t))$ and $A(t) = \int_{h(x)}^{g(x)} f(t)\,dt$

MATH 117

Fall 2021 $\boxed{\#4}$

$\int 0\,dx = C$

$\int k\,dx = kx + C$

$(n \neq -1)$ $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

$(x \neq 0)$ $(n = -1)$ $\int \frac{1}{x}\,dx = \ln|x| + C$

(5.3) Antiderivatives

F is anti of f if $F'(x) = f(x)$.

- General Antiderivative

Let F be an antiderivative... then

$(F(x)+c)' = f(x)$ (on (a, b))

$\int f(x)\,dx = F(x) + C$

Linearity:

$\int (f+g)\,dx = \int f\,dx + \int g\,dx$

$\int cf\,dx = c\int f\,dx$

differential equations:

$\frac{dy}{dx} = f(x)$

unknown:

$y = F(x)$

$du = \frac{du}{dx}dx$ → $du = u'(x)\,dx$

(5.7) Substitution

If $F'(x) = f(x)$, and u is a differentiable function which's range includes the domain of f, then

$$\int f(u(x))\,u'(x)\,dx = F(u(x)) + C$$

$$\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$$

(change of variables formula)

$\boxed{\text{eg}}$ evaluate $\int x\sqrt{5x+1}\,dx$.

① $u = 5x+1$ $\left(x = \frac{1}{5}(u-1)\right)$

② $du = 5dx \Rightarrow \sqrt{5x+1}\,dx = \frac{1}{5}\sqrt{u}\,du$

③ $\int x\sqrt{5x+1}\,dx = \int \left(\frac{1}{5}(u-1)\right)\left(\frac{1}{5}\sqrt{u}\right)du$

④ Solve! $= \frac{2}{125}(5x+1)^{5/2} - \frac{2}{75}(5x+1)^{3/2} + C$

CoVF for DEFINITE INTEGRALS

$$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

$\int_a^b \frac{1}{x}\,dx = \ln\left(\frac{b}{a}\right)$

when a and b have same sign

$\sin x \to -\cos x$

$\cos x \to \sin x$

$\sec^2 x \to \tan x$

$\csc^2 x \to -\cot x$

$\sec x \tan x \to \sec x$

$\csc x \cot x \to -\csc x$

$(k \neq 0)$ $e^{kx} \to \frac{1}{k}e^{kx}$

$\tan x \to \ln|\sec x|$

$\cot x \to \ln|\sin x|$

$\ln x \to x\ln x - x$

$\cosh x \to \sinh x$

$\sinh x \to \cosh x$

$b^x \to \frac{b^x}{\ln(b)}$

$\frac{1}{a^2+x^2} \to \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$\frac{1}{\sqrt{a^2-x^2}} \to \sin^{-1}\left(\frac{x}{a}\right)$

$\frac{1}{x\sqrt{x^2-a^2}} \to \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)$

double-angle...

$\cos(2a) = 1 - 2\sin^2(a)$

(5.8) More special integrals

$\ln x = \int_1^x \frac{dt}{t}$ $(x>0)$ $+C$

$\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$ $(-1<x<1)$ $+C$

other inverse trig functions: see my notes on derivatives.

$f(x) = b^x$

$\int b^x\,dx = \frac{b^x}{\ln(b)} + C$

PARTS example!!

$\int x^2 \cos x\,dx$

$u = x^2$ ↗ $(v = \sin x)$

$dv = \cos x\,dx$

$\int \underbrace{x^2\cos x\,dx}_{u\,dv} = \underbrace{x^2\sin x}_{uv} - \int \underbrace{\sin(x)}_{v}\underbrace{2x\,dx}_{du} + C$

Use integration by parts again...

$u = x$ $dv = \sin x\,dx$

$\int \underbrace{x\sin x\,dx}_{u\,dv} = \underbrace{-x\cos x}_{uv} - \int \underbrace{(-\cos x)}_{v}\underbrace{dx}_{du} = -x\cos x + \sin x + C$

Finally, we combine:

$x^2\sin x + 2x\cos x - 2\sin x + C$

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

(7.1) Integration by Parts

★ from the product rule ★

u(x) and v(x)...

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ ➡ $\int u\,dv = uv - \int v\,du$

main show!

① Choose dv so $v = \int dv$ is evaluatable.

② Choose u so du/dx is simpler than u.

Definite version...

$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

$\sin^7 x$? ⊙⊙ ↓

Reduction Formula!

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

Use double-angle identities for other versions of this ↗

Mobius 5.2

MEAN VALUES

$mv(f) = \frac{1}{b-a}\int_a^b f(x)^2\,dx$

Average Values →

"root mean square value"

$r.m.s(f) = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx}$

Point-slope Form

$m = \frac{y-y_0}{x-x_0}$

$\Rightarrow y = m(x-x_0) + y_0$

(4.7) Applied Optimisation

⓪ Choose variables
① Write the objective function f, and a and b
② Find the critical points of f on $[a, b]$.
③ Consider min/max values of $f(x)$.

(a, b) Closed interval:

consider as the function approaches the endpoint.

(6.1) Area Between Graphs

$$A=\int_a^b (f(x)-g(x))\,dx=\int_a^b (y_{top}-y_{bot})\,dx$$

① Find any points of intersection

② add up the magnitudes of each area!

Integrating y-axis:

right of $x=0$ is positive, left is negative

$$A=\int_c^d (g(y)-h(y))\,dy=\int_c^d (x_{right}-x_{left})\,dy$$

(4.8) Newton's Method

① Start with x_0.
② Use $\boxed{x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}}$.

$\sin^2 x=\frac{1}{2}(1-\cos 2x)$
$\cos^2 x=\frac{1}{2}(1+\cos 2x)$
$\sin 2x=2\sin x\cos x$
$\cos 2x=\cos^2 x-\sin^2 x$

(7.2) Trigonometric Integrals

$\int \sin^2 x\,dx=\frac{x}{2}-\frac{1}{2}\sin x\cos x+C$
$\int \cos^2 x\,dx=\frac{x}{2}+\frac{1}{2}\sin x\cos x+C$
$\int \sin^n x\,dx=-\frac{\sin^{n-1}x\cos x}{n}+\frac{n-1}{n}\int \sin^{n-2}x\,dx$

Integrating $\sin^m x\cos^n x$

- Case 1: m is odd

Split off one $\sin x$, use $\sin^2 x=1-\cos^2 x$.
Then u-sub $u=\cos x$.

- Case 2: n is odd

Split off one $\cos x$, use $\cos^2 x=1-\sin^2 x$
Then u-sub $u=\cos x$

- Case 3: m, n even

1: repeated double-angle formulae
or
2: convert to only sin or cos and use reduction formulas.

SEE PHOTO(S) ON PHONE

If $x=4\tan\theta$:

$\sin\theta=\frac{x}{\sqrt{x^2+16}}$

$\cos\theta=\frac{4}{\sqrt{x^2+16}}$

(7.3) Trigonometric Substitution

We wanna use one of these little guys!
$\sqrt{a^2-x^2}$
$\sqrt{x^2-a^2}$
$\sqrt{x^2+a^2}$
$\sqrt{ax^2+bx+c}$

$\boxed{\sqrt{a^2-x^2} \text{ with } a>0:}$

Assume $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$ and use
$x=a\sin\theta$ $dx=a\cos\theta\,d\theta$ ☆
$\sqrt{a^2-x^2}=a\cos\theta$

$\boxed{\sqrt{x^2+a^2} \text{ with } a>0:}$

Assume $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$, substitute with
$x=a\tan\theta$ $dx=a\sec^2\theta\,d\theta$ $\sqrt{x^2+a^2}=a\sec\theta$

$\theta=\sec^{-1}(\frac{x}{a})$ when $x>a$, but
$\theta=\sec^{-1}(\frac{x}{a})+\frac{\pi}{2}$ when $x<-a$

$\boxed{\sqrt{x^2-a^2} \text{ with } a>0:}$

Assume $0\leq\theta<\frac{\pi}{2}$ when $x\geq a$, and
$\pi\leq\theta<\frac{3\pi}{2}$ when $x\leq -a$, choose:
$x=a\sec\theta$ $dx=a\sec\theta\tan\theta\,d\theta$ $\sqrt{x^2-a^2}=a\tan\theta$

$\boxed{\sqrt{ax^2+bx+c}}$
① Complete the square.
② Trig sub ⍨

Higher powers of n on $(\sqrt{a^2-x^2})^n$ and such:
① Sub to eliminate □ root.
② Evaluate the trig integral.
③ Convert to original variable.

SPECIAL INTEGRATION TECHNIQUES

(6.2) Setting Up Integrals

Volume of body:
sum of areas, like
$\Delta y=\frac{(b-a)}{N\leftarrow\infty}$ and
$V_i\approx A(y_{i-1})\Delta y$, so

$$\boxed{V=\int_a^b A(y)\,dy}$$

① Find Formula for cross-sectional $A(y)$.
② Compute the integral of $A(y)$!

Density: linear mass density is ρ, so:
$\boxed{\text{total mass}=\rho\cdot l}$
density $\rho(x)$...

$$\boxed{M=\int_a^b \rho(x)\,dx}$$

Density: radial density

Population P within radius R $\boxed{P=2\pi\int_0^R r\rho(r)\,dr}$

Flow rate $\boxed{Q=2\pi\int_0^R r\underbrace{v(r)}\,dr}$ ↳ velocity of laminar flow rate

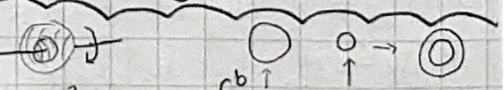
MVT FOR INTEGRALS

if f is continuous on $[a, b]$, then there exists $c\in[a, b]$ such that:

$$\boxed{f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx}$$

(6.3) Disk Method

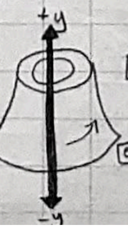
Volume obtained by rotating f from $a\to b$ about the x-axis is:



$$\boxed{V=\pi\int_a^b R^2\,dx=\pi\int_a^b f(x)^2\,dx}$$

If it's a washer-shape: $V=\pi\int_a^b (R_{outer}^2-R_{inner}^2)\,dx=\pi\int_a^b (f(x)^2-g(x)^2)\,dx$

VERTICAL AXIS
① get everything in terms of y.
② choose start/end points of y-values
③ integrate with respect to y



horizontal axis $y=c$

eg $\boxed{R_{outer}=c-g(x)}$ ↳ was inner
IF $c\geq f(x)\geq g(x)$

eg $\boxed{R_{outer}=f(x)-c}$
IF $f(x)\geq g(x)\geq c$

6.4 Cylindrical Shells

Volume of shell $\overset{dV}{\approx} 2\pi R h \underset{\text{thickness}}{\Delta r}$

The solid obtained by rotating the region under $y=f(x)$ over the interval $[a,b]$ about the <u>y-axis</u> has the following volume:

$$\boxed{V = 2\pi \int_a^b \underset{\text{radius}}{x}\, \underset{\text{height of shell}}{f(x)}\, dx} \text{ indeed!}$$

$$\boxed{V = 2\pi \int_a^b x\,(f_{top}(x) - g_{bottom}(x))\,dx}$$

Shell method:
- Find shell <u>height</u>, <u>parallel</u> to axis of rotation.

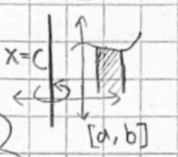
VS

Disk/Washer method:
- Find washer <u>radii</u>, <u>perpendicular</u> to axis of rotation.

<u>integrand</u> based on a new axis:

$c \le a$: $(x-c)f(x)$

$c \ge b$: $(c-x)f(x)$



Horizontal Axis

① "heights" in terms of x
↳ distances on x-axis, now

② Update limits of integration

③ integrate with respect to y.

Theorem 1

p-integral over $[a, \infty)$

for $a>0$:

$$\int_a^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{a^{1-p}}{p-1} & \text{if } p>1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}$$

7.7 Improper Integrals

IMPROPER: (1) Interval of integration is infinite
(2) Integrand tends to infinity anywhere

Definition $\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$

Converge: $\in \mathbb{R}$
Diverge: $\neg \in \mathbb{R}$

(2) Unboundedness:

If the function in the integrand is unbounded on any axis, the integral is improper.

UNBOUNDED = ONE-SIDED

f is continuous on $[a, b)$ and $\lim_{x\to b^-} f(x) = \pm\infty$:

$$\boxed{\int_a^b f(x)\,dx = \lim_{R\to b^-} \int_a^R f(x)\,dx}$$

f is continuous on $(a, b]$ and $\lim_{x\to a^+} f(x) = \pm\infty$:

$$\boxed{\int_a^b f(x)\,dx = \lim_{R\to a^+} \int_R^b f(x)\,dx}$$

Doubly Infinite Integral $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^\infty f(x)\,dx$

True, "assuming both integrals converge"

$$\text{Work} = \int_a^b F(x)\,dx$$

Theorem 2

p-integral over $[0, a]$

for $a>0$

$$\int_0^a \frac{dx}{x^p} = \begin{cases} \dfrac{a^{1-p}}{1-p} & \text{if } p<1 \\ \text{diverges} & \text{if } p \ge 1 \end{cases}$$

Theorem 3

Comparison Test

Assume f, g continuous $f(x) \ge g(x) \ge 0$ for $x \ge a$:
- if $\int_a^\infty f(x)\,dx$ converges, $\int_a^\infty g(x)\,dx$ converges
- if $\int_a^\infty g(x)\,dx$ diverges, $\int_a^\infty f(x)\,dx$ diverges

8.2 Arc Length & Surface Area

Arc Length ⓢ = polygonal approximation lengths $|L|$ as $\|P\| \to 0$

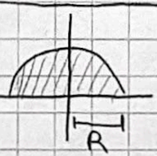
$$\boxed{s = \lim_{\|P\|\to 0} \sum_{i=1}^{N} |L_i|} \qquad \boxed{s = \int_a^b \sqrt{1+f'(x)^2}\,dx}$$

Surface Area

Area of surface of revolution:

$$S = 2\pi \int_a^b f(x)\sqrt{1+f'(x)}\,dx$$

Graph of Semicircle:



$\Rightarrow f(x) = \sqrt{R^2 - x^2}$

Differentials $\Delta y \approx \frac{dy}{dx}\Delta x$ $\quad f(x_f) = f(x_0 + \Delta x) \approx \boxed{f(x_0) + f'(x_0)\Delta x}$

they are cute and innocent, confused heading into their 117 final exam