MATH 115 - F21
Linear Algebra
Full Course Notes

With Prof Kwan Tsaan Lai

These notes are exhaustive. If you're not in SE (taking MATH 135), I recommend learning elementary proof techniques.

Josiah Plett

MATH 115 FALL 2021 $\boxed{\#1}$

☆ $0 \le \theta \le 2\pi$ ☆

① Complex Numbers

$\boxed{\text{A complex number in standard form: } \boxed{x+yj}}$

$\mathbb{N}$ = Natural Numbers {1 and up}
$\mathbb{Z}$ = Integers
$\mathbb{Q}$ = Rational Numbers $\{\frac{a}{b} \mid a,b \in \mathbb{Z}, b \neq 0\}$
$\mathbb{R}$ = Real Numbers {all $\mathbb{Q}$ and irrationals}
$\mathbb{C}$ = Complex Numbers $\{x+yj \mid x,y \in \mathbb{R}\}$

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ $\quad \subseteq$ means "is a subset of"

Real part, Imaginary part

if $z = x+yj \in \mathbb{C}$, then $\boxed{x = Re(z)}$ and $\boxed{y = Im(z) \in \mathbb{R}}$

+ − To add or subtract, simply do the Re part separate from the Im part.
× To multiply, simply use binomial expansion along with $j^2 = -1$.
÷ $\boxed{(x+yj)(x-yj) = x^2+y^2 \in \mathbb{R}}$ To divide, use □ to realize the denominator.

② Algebra (in the world of $\mathbb{C}$)

To solve all $z \in \mathbb{C}$ satisfying $z^2 = -7+24j$:
① let $z = a+bj$, then simplify
② equate real and imaginary parts
③ solve the system of equations

"complex conjugates"

Conjugates of complexes: $\boxed{\text{if } z = x+yj,\ \xrightarrow{\text{conjugate}} \bar{z} = x-yj.}$ $\boxed{\text{eg}}$ $\overline{\sqrt{2}j} = -\sqrt{2}j$ or $\overline{-4} = -4$

Properties of complex conjugates:

$\overline{z+w} = \bar{z}+\bar{w}$
$\overline{z^k} = \bar{z}^k$ for $k \in \mathbb{Z}$, $k \ge 0$, ($k \neq 0$ if $z = 0$)
$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$ for $w \neq 0$ $\quad \overline{zw} = \bar{z}\bar{w}$
$z + \bar{z} = 2x = 2Re(z)$
$z - \bar{z} = 2yj = 2jIm(z)$
$z\bar{z} = x^2+y^2$ main!

$z = x+yj$

% Modulus of complexes

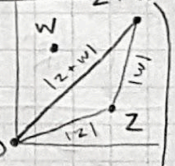
absolute value $\boxed{|z| = \sqrt{x^2+y^2}} = \sqrt{z\bar{z}}$

Properties of modulus:

$|\bar{z}| = |z|$
$z\bar{z} = |z|^2$
$|z+w| \le |z|+|w|$ "Triangle Identity"

Geometrical Thinking: $\boxed{\bar{z}}$ is a reflection of z in the x-axis. (Re)
$\boxed{|z|}$ is the distance between 0 and z.

addition — Triangle Inequality:



$|z+w| \le |z|+|w|$

③ Polar Form (of Complex Numbers)

$z = r(\cos\theta + j\sin\theta)$

radius → $r = |z|$ (for non-zero z)
argument → θ = angle z makes w/origin

HELPS WITH COMPLEX MULTIPLICATION:

$z_1z_2 = r_1r_2(\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2))$

☆angle sum formulae☆
$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$
$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$

Computing θ (argument):
$\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$
from which we obtain
$z = r(\cos\theta + j\sin\theta)$

HELPS EVEN MORE WITH COMPLEX POWERS: $\boxed{n \in \mathbb{Z}}$

$z^n = r^n(\cos(n\theta) + j\sin(n\theta))$ ← De Moivre's Theorem

for $P(w) = 3w^3+2w^2+4w+1$,
$P(w) = 3(w-z)(w-\bar{z})(w-y)$ another root

if z is a root of $p(w)$, then $w-z$ and $w-\bar{z}$ are factors, so we could have

④ Roots & Exponentials

Finding the n^{th} root of z: $(k \in \mathbb{Z})$ $(k \leq n-1)$

$$\boxed{\left(\sqrt[n]{z}\right)_k = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + j\sin\left(\frac{\theta+2k\pi}{n}\right)\right)}$$

n^{th} root of z, but exponential:

$$\boxed{\left(\sqrt[n]{z}\right)_k = r^{\frac{1}{n}}\left(e^{j\frac{\theta+2k\pi}{n}}\right)}$$

Complex Exponential Form

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\boxed{z = re^{j\theta}}$$

Follows all the same multiplicative and exponential algebraic laws as numbers do!

Euler's Identity: $\boxed{e^{j\pi} + 1 = 0}$

WEEK1 (SEPT 8-10) COMPLETE!

⑤ Complex Polynomials

if z is a root of <u>REAL</u> polynomial $p(x)$, so too is complex conjugate $\bar{z}$. $(p(\bar{z}) = 0)$

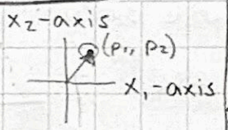
To solve complex polynomials, use algebra and the quadratic formula, as usual.

REFER TO LECTURE ⑤ FOR SOME SOLID PROOF EXAMPLES

By the way, try to <u>avoid roundoff error!</u>

⑥ Vector Algebra

Multi-Dimensional Vectors $\mathbb{R}^n \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \, x_1, \dots, x_n \in \mathbb{R} \right\}$

$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \Rightarrow$ 

$\mathbb{R}^2$ is all such vectors

$\mathbb{C} \neq \mathbb{R} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, x_1, x_2 \in \mathbb{R} \right\}$

Vectors are only equal if $x_1 = y_1, \dots x_n = y_n$; otherwise, $\boxed{\vec{x} \neq \vec{y}}$

$$\vec{0}_{\mathbb{R}^n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Parallel Vectors
- must be scalar multiples of each other.

VECTOR ADDITION

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \in \mathbb{R}^n$$

SCALAR MULTIPLICATION

$$c\vec{x} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix} \in \mathbb{R}^n$$

Given a point p...

$\vec{p} = \overrightarrow{Op}$ (O: origin)

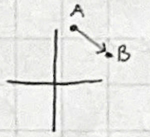
→ a position vector, in <u>standard</u> position.

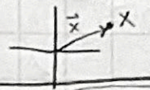
Linear Combination (of vectors $\vec{x}_1$ through $\vec{x}_k$)

$\vec{x}_1, \dots \vec{x}_k \in \mathbb{R}^n$ and $c_1 \dots c_k \in \mathbb{R}$ for $k \in \mathbb{N}$:

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_k\vec{x}_k$$

($\therefore$ $\mathbb{R}^n$ is <u>closed under</u> linear combinations)

$\overrightarrow{AB} \Rightarrow$ 

$\vec{x} \Rightarrow$ 

final minus initial

$$\overrightarrow{AB} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \end{bmatrix} = \overrightarrow{OB} - \overrightarrow{OA}$$

$\vec{y} = \frac{1}{\|\vec{x}\|}\vec{x}$ $\therefore$ $\vec{y}$ is a <u>UNIT VECTOR</u> in the direction of $\vec{x}$.

⑦ Norms & Dot Products

(magnitude)

Norm: (non-negative real number) $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ — properties →
- $\|c\vec{x}\| = |c| \cdot \|\vec{x}\|$
- $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

$\|\vec{x}\| = 1$; $\vec{x}$ is a <u>unit vector</u>

Dot Product: (real number) $\vec{x} \cdot \vec{y} = x_1y_1 + \dots + x_ny_n$ — properties →
- $\vec{x} \cdot \vec{y} \in \mathbb{R}$ (distributive law? YES.)
- $\vec{x} \cdot \vec{x} = \|\vec{x}\|^2$
- $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$

Angle: $\vec{x} \cdot \vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\theta$

- acute: $\vec{x} \cdot \vec{y} > 0$
- orthogonal: $\vec{x} \cdot \vec{y} = 0$
- obtuse: $\vec{x} \cdot \vec{y} < 0$

thinking about $\cos\theta$

The zero vector $\vec{0}_{\mathbb{R}^n}$ is <u>orthogonal</u> to all other $\vec{x} \in \mathbb{R}^n$!

Cosine law!

$$\|\vec{x} - \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$$

$$\|\vec{x} - \vec{y}\|^2 = (\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y})$$
$$= \|\vec{x}\|^2 - 2(\vec{x} \cdot \vec{y}) + \|\vec{y}\|^2$$

$$\langle \alpha\vec{z}, \vec{w} \rangle = \alpha\langle \vec{z}, \vec{w} \rangle$$
$$\langle \vec{z}, \alpha\vec{w} \rangle = \bar{\alpha}\langle \vec{z}, \vec{w} \rangle$$

any degree ⓝ polynomial can be written as:

$$p(z) = k(z - w_1)(z - w_2)\dots(z - w_n)$$

solving complex polynomials

AREA of PARALLELOGRAM (in $\mathbb{R}^3$) $= \|\vec{x} \times \vec{y}\|$

$$\vec{v} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{v} \times \vec{w}) = 0$$

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$|\langle\vec{w},\vec{z}\rangle| \leq \|\vec{w}\|\|\vec{z}\|$ → Cauchy-Schwarz Inequality

$\|\vec{z}+\vec{w}\| \leq \|\vec{z}\| + \|\vec{w}\|$ → triangle inequality

8 Complex Vectors

$$\mathbb{C}^n = \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \middle| \; z_1, \dots z_n \in \mathbb{C} \right\}$$

Fancy way of saying Complex dot product

fact: $\langle \vec{x}, \vec{x} \rangle \in \mathbb{R}$ ≥ 0

$\langle c\vec{z}, d\vec{z} \rangle = \bar{c}\, d \langle \vec{z}, \vec{z} \rangle$

Vector addition: add each term.

Scalar multiplication: multiply each term.

complex conjugates!

Complex inner product: $\langle \vec{x}, \vec{y} \rangle = \bar{x}_1 y_1 + \dots + \bar{x}_n y_n \in \mathbb{C}$.

Norm: $\sqrt{\text{complex inner product}} = \|\vec{z}\| = \sqrt{\langle \vec{z}, \vec{z} \rangle}$

Conjugate: conjugate of each term.

Dot Product: $\bar{\vec{x}} \cdot \vec{y} = \langle \vec{x}, \vec{y} \rangle \in \mathbb{C}$.

conjugate!

$\vec{0}_{\mathbb{R}^n}$ is orthogonal to all $\vec{x} \in \mathbb{R}^n$!

$\langle \alpha \vec{z}, \vec{z} \rangle = \bar{\alpha} \langle \vec{z}, \vec{z} \rangle$

$\langle \vec{z}, \alpha \vec{z} \rangle = \alpha \langle \vec{z}, \vec{z} \rangle$

Properties of complex inner products

Not all are here!

$\langle \vec{z}, \vec{z} \rangle \geq 0$

$\langle \vec{z}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{z} \rangle}$

$\langle \vec{v}+\vec{z}, \vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle + \langle \vec{z}, \vec{w} \rangle$

$|\langle \vec{z}, \vec{w} \rangle| \leq \|\vec{z}\|\|\vec{w}\|$

← distributive laws

"Vector Product"

Cross Product in $\mathbb{R}^3$

⇨ NOT VALID IN ANY OTHER $\mathbb{R}^n$ SETS (except $\mathbb{R}^7$)

$$\vec{x} \times \vec{y} = \begin{bmatrix} x_2y_3 - y_2x_3 \\ -(x_1y_3 - y_1x_3) \\ x_1y_2 - y_1x_2 \end{bmatrix}$$

★ The cross product is orthogonal to both $\vec{x}$ and $\vec{y}$

CROSS PRODUCT TRICK

determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{array}{c|ccccc|c} x_1 & x_2 & x_3 & x_1 & x_2 & & x_3 \\ y_1 & y_2 & y_3 & y_1 & y_2 & & y_3 \end{array}$$

Z_1, Z_2, Z_3

$$\vec{x} \times \vec{y} = \vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

9 Properties of Cross Products (working in $\mathbb{R}^3$)

$\vec{x} \times \vec{y} = -(\vec{y} \times \vec{x})$ → anticommutativity

Cross-product of parallel vectors is always $\vec{0}$.

for $\mathbb{R}^3$...

Lagrange Identity ⇨ $\|\vec{x} \times \vec{y}\|^2 = \|\vec{x}\|^2 \|\vec{y}\|^2 - (\vec{x} \cdot \vec{y})^2$

$\vec{x} \cdot \vec{y} = \|\vec{x}\|\|\vec{y}\| \cos\theta$

$\|\vec{x} \times \vec{y}\| = \|\vec{x}\|\|\vec{y}\| \sin\theta$

Length of the cross product is the area of the parallelogram.

9.65

VECTOR EQUATION OF A LINE (vector equation)

line in $\mathbb{R}^n$ through point P with non-zero direction $\vec{d}$ is:

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{OP} + c\vec{d}, \; c \in \mathbb{R}$$

when $A(1,1,-1)$ and $B(4,0,3)$

Parametric Equations of the line → $\vec{x}$ →

$x_1 = 1+3t$

$x_2 = 1-t$ $\quad t \in \mathbb{R}$

$x_3 = -1-2t$

10 Vector Equations of Planes

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{OP} + s\underbrace{\vec{u} + t\vec{v}}_{\text{non-zero, non-parallel}}, \; s, t \in \mathbb{R}$$

★ Planes are parallel if their normal vectors are parallel ★ (in $\mathbb{R}^3$)

When given a line equation and a point, take the direction vector of the line, and make another direction vector from a point on the line to your given point! And the initial vector will be your given point too.

of planes

Normal Vectors (in $\mathbb{R}^3$)

$\vec{n}$ is the unique line through P such that for any Q on the plane, $\vec{n}$ is orthogonal to $\vec{PQ}$.

Scalar Equations of a Plane in $\mathbb{R}^3$

With $P(a,b,c)$ being a point on the plane, we know any point $Q(x_1, x_2, x_3)$ is also on the plane if:

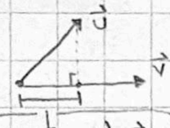
normal vector

$$0 = \vec{n} \cdot \vec{PQ} = \vec{n} \cdot (\vec{OQ} - \vec{OP}) = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 - a \\ x_2 - b \\ x_3 - c \end{bmatrix} = n_1(x_1-a) + n_2(x_2-b) + n_3(x_3-c)$$

$\therefore n_1x_1 + n_2x_2 + n_3x_3 = n_1a + n_2b + n_3c$ ← SCALAR EQUATION

When you get plane T: $c_1x_1 + c_2x_2 + c_3x_3 = y$, the $\vec{n}$ to T is (c_1, c_2, c_3)!!

(11) Projections



$[\,$ $\text{perp}_{\vec{v}}\vec{u} = \vec{u} - \text{proj}_{\vec{v}}\vec{u}$

Try plugging in zeros!!

$\vec{v}\,\vec{u} \rightarrow \boxed{\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}} \rightarrow \text{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$

Strategy: Distance P is from Plane T / Line L etc..

1. Determine an arbitrary point P_0 on T/L.
2. Project $\overrightarrow{P_0P}$ into $\text{perp}_{\vec{n}/\vec{d}}\, \overrightarrow{P_0P}$.
3. The norm of part (2) is the minimum distance.
4. Find $\vec{Q}$ with $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$ since $\overrightarrow{PQ} = -\text{perp}_{\text{stuff}}\overrightarrow{P_0P}$

(12) Volumes of Parallelepipeds (in $\mathbb{R}^3$)

3 non-zero, non-parallel, non-plane-sharing (plane determined by the other two vectors) vectors form a 3-dimensional parallelogram.

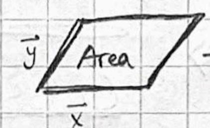
$$V = \|\vec{x}\times\vec{y}\|\,\|\text{proj}_{\vec{x}\times\vec{y}}\vec{w}\| = \boxed{|\vec{w}\cdot(\vec{x}\times\vec{y})|}$$

but it could be any of the sides, so we also have:

$= |\vec{w}\cdot(\vec{y}\times\vec{x})| = |\vec{y}\cdot(\vec{x}\times\vec{w})| = |\vec{y}\cdot(\vec{w}\times\vec{x})|$
$= |\vec{x}\cdot(\vec{w}\times\vec{y})| = |\vec{x}\cdot(\vec{y}\times\vec{w})|$

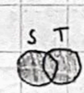
The formula holds for any $\vec{x},\vec{y},\vec{w} \in \mathbb{R}^3$!

(12.2) SET THEORY Intro

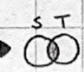


$\rightarrow \|\vec{x}\times\vec{y}\|$

- Set: collection of objects. $x \in S \rightarrow$ x is an element of set S.
- Element: object of a set. $x \notin \rightarrow$ not that ☺
- Union: $\cup$ or "or" $\rightarrow \boxed{S\cup T} = \{x \mid x\in S \text{ or } x \in T\}$ →



- Intersection: $\cap$ or "and" $\rightarrow \boxed{S\cap T} = \{x \mid x \in S \text{ and } x \in T\}$ →



- Subset: S is a subset of T if $x\in S \Rightarrow x \in T$; $\boxed{S\subseteq T}$
- Empty Set: $\emptyset$ ... fun fact: $\boxed{\emptyset \subseteq S}$ for all sets S! → This is "vacuously true" because it cannot be false.
- Equivalency: $\boxed{S=T}$ if $S\subseteq T$ and $T \subseteq S$.

(13) Spanning Sets

[eg] Let $B = \{v_1, \dots v_k\}$ be a set of vectors in $\mathbb{R}^n$.
The span of B is $\{c_1\vec{v}_1 + \dots + c_k\vec{v}_k \mid c_1, \dots c_k \in \mathbb{R}\} = \text{Span}B$

The set *SpanB* is *spanned* by B, and B is a *spanning set* for SpanB.

$\boxed{B \subseteq \text{Span}B}$ everywhere in $\mathbb{R}^3$...

☆ SpanB is the set of all linear combinations of B's vectors. (non-parallel)

– Geometrical interpretation of spans: with (1) element, line. with (2) elements, plane. with (3), 3-dimensional!

☆ If $\vec{v}_i$ in a span (call it "S") is a linear combination of other $\vec{v}_k$ in S, than we can essentially "remove" $\vec{v}_i$.

(14) Linear Dependence and Linear Independence

Given many vectors with high dimensions, it's hard to tell whether one element is a linear combination of the others, so now we define:

Let $B = \{\vec{v}_1, \dots, \vec{v}_k\}$

> B is linearly dependent if there exist non-zero c's so that $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$
>
> B is linearly independent if the only solution to $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$ is all c's $= \vec{0}$.

[eg] Show that SpanB is linearly independent.

Solution:
1. obtain system of equations w/ c_n.
2. solve the system for trivial c's.
3. state the type of trivial solution we have obtained, then conclude.

→ Write concluding sentences. (show the mathematical contradictions)

EXAMPLES OF PROOFS in lecture 14 ☺

→ ALL subsets of a linearly independent set are themselves linearly independent.

(15) Bases*

*plural of 'Basis'

Let S be a subset of $\mathbb{R}^n$. If B is linearly independent set of $\vec{v}$ from S such that $S = \text{span}B$, then B is a basis for S.

Proving that $B = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$

1. Show that $\text{Span}B = \mathbb{R}^2$, by:
   - (A) For $\vec{x}$ in $\mathbb{R}^2$, show that $\vec{x} \in \text{Span}B$ (so $\mathbb{R}^2 \subseteq \text{Span}B$)
   - (B) Explain $\text{Span}B \subseteq \mathbb{R}^2$ with "closed under linear combinations".
2. Show that B is linearly independent ($\begin{bmatrix}0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\0\end{bmatrix} \ldots$)

Standard Basis: For $i = 1, 2, \ldots n$, $\vec{e}_i \in \mathbb{R}^n$ whose i'th entry is 1 and others are zero; so, the set $\{\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\}$ is the standard basis for $\mathbb{R}^n$.

★ If B is a basis for $S \subseteq \mathbb{R}^n$, then every $\vec{x} \in S$ can be expressed as a unique linear combination of $v_k \in B$ ★

B is then a minimally spanning set for S.

★ A subset S of $\mathbb{R}^n$ is only a $\boxed{\text{subspace}}$ of $\mathbb{R}^n$ if the rules of $\mathbb{R}^n$ hold for it too: ★

1. closed under addition
2. addition is commutative ($x+y=y+x$) & associative ($(x+y)+w = x+(y+w)$)
3. There's a zero vector for all $\vec{x} \in S$ in S!
4. $-\vec{x}$ is an additive inverse in S
5. Multiplication: closed under; is associative; distributive law, and 1 = multiplicative identity.

Trivial Subspace $= \{\vec{0}\}$ (Span $\emptyset = \{\vec{0}\}$)

(16) Subspace Tests

Test for S: If for every $\vec{x}, \vec{y} \in S$ and for every $c \in \mathbb{R}$, we have that $\vec{x} + \vec{y} \in S$ and $c\vec{x} \in S$, then S is a subspace of $\mathbb{R}^n$.

Proving that $S = \left\{\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} \middle| \text{stuff}\right\}$ is a subspace of $\mathbb{R}^n$

1. Show that $\vec{0} \in S$ (because otherwise, it's not a subspace)

   1.5 $S \subseteq \mathbb{R}^n$ where n is defined.
2. Let $\vec{x}, \vec{y} \in S$ and say "there exist $c_1, c_2 \in \mathbb{R}$ such that [to actually define $\vec{x}$ & $\vec{y}$ algebraically]."
3. Use (2) to prove $\vec{x} + \vec{y} \in S$ and $c\vec{x} \in S$, $c \in \mathbb{R}$.

• Fact: every subspace S of $\mathbb{R}^n$ can be expressed as $S = \text{Span}\{v_1, v_2, \ldots v_k\}$.

Now, to find such a Basis of S:

1. Choose an arbitrary $\vec{x} \in S$ and use algebra to "decompose" $\vec{x}$ into a linear combination of vectors that are inside S.
2. Use (1) to create a full $\text{Span}\{\text{vector}, \text{vector}, \ldots\}$
3. Get B as a linearly independent version of that.

(17) K-Flats and Hyperplanes

Definition of lines + planes (generalized): For a positive int $k \leq n-1$, let $\vec{v}_1, \ldots \vec{v}_k \in \mathbb{R}^n$ be such that $\{\vec{v}_1, \ldots \vec{v}_k\}$ is linearly independant. A $\boxed{\text{k-Flat in } \mathbb{R}^n}$ through p is:

$$\boxed{\vec{x} = \vec{p} + c_1\vec{v}_1 + \ldots + c_k\vec{v}_k}$$

Hyper-Planes: K-flats with $\boxed{k = n-1}$ (A 4-Flat is a hyperplane in $\mathbb{R}^5$)

Hyperplanes are the ONLY K-Flats in $\mathbb{R}^n$ with SCALAR EQUATIONS

ALL SUBSPACES OF $\mathbb{R}^n$ are the K-flats through the ORIGIN

Orthogonal Set: If $\vec{v}_i \cdot \vec{v}_f = 0$ for all $i \neq f$, it's an orthogonal set.
(Orthogonal sets without $\vec{0}$ are always linearly independant.)

Orthogonal Bases: An orthogonal set $\{v_1, v_2, \ldots v_k\}$ that is a basis of a subspace S of $\mathbb{R}^n$.

Any $\vec{x} \in S$ can be expressed as $\vec{x} = c_1\vec{v}_1 + \ldots + c_k\vec{v}_k$ where $c_i = \dfrac{\vec{x} \cdot \vec{v}_i}{\|v_i\|^2}$

Orthonormal means that for every $\vec{v}_i$, $\|v_i\| = 1$

If Orthogonal & Orthonormal: $c_i = \vec{x} \cdot \vec{v}_i$

Finding a Basis for a Subspace: (1) Let $\vec{x} \in S$. Define $\vec{x}$'s properties based on S's restrictions. (2) Use basic vector algebra to write $\vec{x}$ as a linear combination of some vectors; those are your bases!

((1) Arbitrary $\vec{x} \in S$, (2) "decompose" into linear combo.)

HOMOGENOUS SYSTEM: Bunch of linear equations whose only solution is $\vec{0}$.

(18) Systems of Linear Equations

A vector $\vec{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \in \mathbb{R}^n$ is a solution to a system of (m) equations in (n) variables if all (m) equations are satisfied when we set $x_j = s_j$ for $1 \le j \le n$. The set of all such solutions is called a *solution set*.

Parametric: $x_1 = 5$, $x_2 = 7$

(Forms)

Vector: $\vec{x} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Point form: $(5, 7)$

Example Linear system:

$x_1 - x_2 = -2$

$4x_1 - 3x_2 = -1$

Coefficient MATRIX: $A = \begin{bmatrix} 1 & -1 \\ 4 & 3 \end{bmatrix}$

Constant MATRIX: $\vec{b} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

Augmented MATRIX: $[A|\vec{b}] = \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 4 & 3 & 3 \end{array}\right]$

"Equivalent" means having the exact same solution set.

ERO's (Elementary Row Operations) on Matrices

- Swap two rows
- Add a scalar multiple of one row to another
- Multiply any row by a *non-zero* scalar

Denoted By:

$R_i \leftrightarrow R_j$

$R_i + R_j$ (written at R_i)

$7R_i$ (written at R_i)

☆ Back Substitution ☆

The first time you get a solution to a variable (like $0\,0\,1\,0 | 7$), use algebraic subbing to finish/verify.

(19) Solving linear systems of equations

A "Leading Entry" is the first non-zero entry in each row of a matrix.

REF Row Echelon Form: (not a unique form)

(1) All rows containing non-zero entries appear first (above).

(2) Each leading entry is to the right of all previous leading entries.

RREF Reduced Row Echelon Form: (always Uniqe)

(3) Each leading entry is a 1 (called a "leading one")

(4) Each leading one is the only non-zero entry in its column.

When you've got a row of 0's: Use the REF to make a system of equations and then solve those into the vector form of a line or plane or hyperplane or whatever.

$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$

a_{ij} : coefficient on x_j ; equation i.

represents x_1

represents x_2

x_3

represents the constant of the equation.

A "Leading Variable" is x_j when the column j contains a leading entry.

A "Free Variable" is x_j when the column j has no leading entry.

To solve when there is a free variable, set $x_j = \underbrace{t \in \mathbb{R}}_{\text{unique for each.}}$ and then make a vector equation!

If you have something like $[0\,0\,0|4]$, then the system inconsistent; ∴ it has no solutions.

THIS ALL WORKS WITH COMPLEX NUMBERS TOO.

A system has any solutions $\iff$ the system is consistent ☆

(20) MORE MATRIX STUFF

eg: $\operatorname{rank}\left(\begin{bmatrix} 6 & 3 \\ -2 & -1 \end{bmatrix}\right) = 1$

Let $\vec{v}_1, \dots \vec{v}_k \in \mathbb{R}^n$. Then $\vec{b} \in \operatorname{Span}\{\vec{v}_1, \dots \vec{v}_k\} \iff [\vec{v}_1\ \vec{v}_2 \dots \vec{v}_k | \vec{b}]$ is consistent. (An augemented matrix with (n) rows)

A "Rank" of a matrix A, denoted by rank(A), is the number of leading entries in an REF of A.

If a Matrix has m rows & n columns, $\operatorname{rank}(A) \le \min\{m, n\}$

System-Rank Theorem

$[A|\vec{b}]$ be the augemented matrix of this system: "m" linear equations and "n" variables.

(1) System is consistent if & only if $\operatorname{rank}(A) = \operatorname{rank}([A|\vec{b}])$.

(2) If (is_consistent): # of parameters in general solution $= n - \operatorname{rank}(A)$.

(3) System is $\underbrace{\text{consistent}}_{\text{INCONSISTENT: take negation}}$ for all $\vec{b} \in \mathbb{R}^m$ if & only if $\operatorname{rank}(A) = m$.

MATH 115

Fall 2021 #4

21 Homogeneous Systems

Theorem 21.4: A consistent, underdetermined system has infinitely many real solutions.

System-Rank Theorem Recap: $\text{Rank}(A) = \text{Rank}([A|\vec{b}])$ then system is consistent

- Underdetermined: If $n > m$; that is, more variables than equations.
- Overdetermined: If $\boxed{m > n}$; that is, more equations than variables.
- Homogeneous linear equation: equation with constant term of zero.
  - ALWAYS CONSISTENT ↔ ONLY NEED COEFFICIENT MATRIX
- Associated Homogeneous System: Given augmented matrix $[A|\vec{b}]$, AHS is $[A|\vec{0}]$.

Actually Solving Stuff → Solution to Homogeneous System is a k-Flat through the origin; The particular solution to a related augemented matrix is the addition of a point!

Theorem 21.12: Any solution set of a Homogeneous System is a Subspace of $\mathbb{R}^n$

22 H.S. and Bases and Such

Consider AM $[A|\vec{b}]$: with m linear equations and n variables:

If $\text{rank}(A) = k < n$, then the solution is of the form
$$\vec{x} = \vec{d} + t_1\vec{v}_1 + \dots + t_{n-k}\vec{v}_{n-k}$$
with $\{\vec{v}_1, \dots \vec{v}_{n-k}\}$ being linearly independent.

LINEAR INDEPENDENCE: Given an arbitrary spanning set, we can check if it is linearly independent by carrying its coeffient matrix to RREF! Then remove any vectors that correspond to any free variables!

Say we have: $\begin{bmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \\ w_3 & x_3 & y_3 & z_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Then $\vec{y}$ can be written as:

$\vec{y} = -\vec{w} + 2\vec{x}$ since $\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

AND:

$\vec{z} = 3\vec{x}$ since $\left[\begin{array}{c|c} 1 & 0 \\ 0 & 3 \\ 0 & 0 \end{array}\right]$

Theorem 22.3 — Dependence of Bases → Let $B = \{\vec{v}_1 \dots \vec{v}_k\}$ be a basis for a subspace S in $\mathbb{R}^n$. If $C = \{\vec{w}_1 \dots \vec{w}_\ell\}$ is a set in S with $\boxed{\ell > k}$, than C must be linearly dependent.

Any basis B which is a subspace of S in $\mathbb{R}^n$ has $\boxed{\text{the same \# of vectors}}$ as any other such basis

- Dimension: If B is a basis for subspace S of $\mathbb{R}$, then $\dim(S) = k$, k is # of vectors in B.

If S is k-dimensional subspace of $\mathbb{R}^n$ with $k > 0$:
(1) Sets with more than k vectors in S are L.I.
(2) Sets with fewer than k vectors can't span S.
(3) Set w/ k vectors Spans S ⟺ it's L.I.

23 Applications!

Chemical Reactions: let $x_1 \dots x_n$ represent the # of each molecule, and arrange a homogeneous system by equating each element!

Linear Models: Set variables to the things that are varying, get the RREF, let the free variables free and use big brain to finish. :

24 Applications! (again)

Network Flow: Junction rule: $\boxed{\text{flow-in} = \text{flow-out}}$

Use JR on every node to get the system of linear equations

Electrical Networks:
- ▲ Voltage: from − to +, +V, otherwise −V.
- ▲ Ohm's Law: $V = IR$ across a resistor.
- ▲ Kirchoff's Laws: closed voltage loop sum = 0. ← CE
- ▲ Kirchoff's Laws 2: current-in = current-out. ← CC

applies to reference current too!! Only need to consider "smallest" loops.

$(A)_{ij} = a_{ij}$ (specific entry)

$3\times2 \to y \times x$!

25 MATRIX ALGEBRA

m rows, n columns: $\begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix} = A$

All $m \times n$ matrices in $\mathbb{R}$ are: $M_{m\times n}(\mathbb{R})$

Zero Matrix is $O_{m\times n}$ or O

- Addition: $(A+B)_{ij} = (A)_{ij} + (B)_{ij}$
- Scalar Multiplication: $(cA)_{ij} = c(A)_{ij}$ $(c \in \mathbb{R})$

$A, B, C \in M_{m\times n}(\mathbb{R})$ & $c, d \in \mathbb{R}$:

1. $A + B \in M(\mathbb{R})$
2. Addition is commutative and associative.
3. $\exists \vec{0}$ st $A + O = A$
4. $(-A) + (A) = O$
5. $cA \in M(\mathbb{R})$
6. for scalar multiplication we have:
   - distributive law
   - associativity
7. Scalar multiplicative identity is 1.

(25) Continued...

- Transpose (A^T): $(A^T)_{ij} = (A)_{ji}$

1. $(A^T)^T = A$
2. $A^T \in M_{n\times m}(\mathbb{R})$
3. $(A+B)^T = A^T + B^T$
4. $(cA)^T = c(A)^T$

- Symmetry: $\boxed{A^T = A}$ $\downarrow$ A is square; $m=n$

1. $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
2. $(A+B)\vec{x} = A\vec{x} + B\vec{x}$
3. $A(c\vec{x}) = c(A\vec{x}) = (cA)\vec{x}$

$\underbrace{A}_{m\times n}\ \underbrace{\vec{x}}_{\mathbb{R}^n} = \underbrace{\vec{b}}_{\mathbb{R}^m}$

(26) The Matrix-Vector Product ☆☆☆☆☆☆☆☆☆

Let $A = [\vec{a}_1 \cdots \vec{a}_n] \in M_{m\times n}(\mathbb{R})$ (so $\vec{a}_1 \ldots \vec{a}_n \in \mathbb{R}^m$) and $\vec{x} = [x_1 \cdots x_n]^T \in \mathbb{R}^n$. Then $A\vec{x}$ is:

$$\boxed{A\vec{x} = x_1\vec{a}_1 + \ldots + x_n\vec{a}_n \in \mathbb{R}^m}$$

THEOREM 26.3

1. Every linear system can be expressed as $A\vec{x} = \vec{b}$.
2. $A\vec{x} = \vec{b}$ is consistent $\Leftrightarrow$ $\vec{b}$ is a linear combination of the colums of A.
3. Using the above definitions, $\vec{x}$ satisfies $A\vec{x} = \vec{b} \Leftrightarrow x_1\vec{a}_1 + \ldots + x_n\vec{a}_n = \vec{b}$

Given $A \in M_{m\times n}(\mathbb{R})$, $\exists \vec{r}_1, \ldots \vec{r}_m \in \mathbb{R}^n$ such that:

$$A = \begin{bmatrix} \vec{r}_1^T \\ \vdots \\ \vec{r}_m^T \end{bmatrix} \xrightarrow{\forall \vec{x} \in \mathbb{R}^n} A\vec{x} = \begin{bmatrix} \vec{r}_1^T \\ \vdots \\ \vec{r}_m^T \end{bmatrix}\vec{x} = \boxed{\begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix}}$$

Identity Matrix

The $n\times n$ identity matrix, I_n, is the square matrix of size $n\times n$ with $a_{ii} = 1$, $i \in \{1, 2, \ldots n\}$ and other entries being zero. $\to \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix} = I_3$

$I_2 = \begin{bmatrix} 1&0\\0&1 \end{bmatrix} = [\vec{e}_1, \vec{e}_2]$ $\boxed{I_n\vec{x} = \vec{x}}$, $\vec{x} \in \mathbb{R}^n$

- Matrices Equal Theorem:

☆ If $A\vec{x} = B\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$, then $A = B$.

$\boxed{A\vec{x} = \vec{b} \text{ is consistent}} \Longleftrightarrow \boxed{\vec{b} \text{ linear combo of columns of } A}$

ALL OF THIS HOLDS FOR COMPLEX MATRICES TOO!! dot product, not cip: $\langle a,b \rangle$

(27) Fundamental Subspaces of a Matrix

1. $\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$ is the $\boxed{\text{nullspace}}$ (or kernel)
2. $\text{Col}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\} = \text{Span}\{\vec{a}_1, \cdots \vec{a}_n\}$ for $A = [\vec{a}_1 \cdots \vec{a}_n] \in M_{m\times n}(\mathbb{R})$ is the $\boxed{\text{column space}}$
3. $\text{ROW}(A) = \{A^T\vec{x} \mid \vec{x} \in \mathbb{R}^m\} = \text{Span}\{\vec{r}_1, \ldots \vec{r}_m\}$ for $A^T = [\vec{r}_1 \ldots \vec{r}_m] \in M_{n\times m}(\mathbb{R})$ is the $\boxed{\text{row space}}$

Th. 27.4 If R is obtained by performing elementary row operations on A, then $\text{Row}(R) = \text{Row}(A)$

$A\vec{x} = \vec{0}$ — so it's basis. (dim(n-rank))

Finding Basis For:

- Null(A): Carry A to RREF, and the vector equation's direction vectors will span the kernel.
- Col(A): The columns in the RREF of A with leading entries will span col(A) independently.
- Row(A): The rows of the RREF of A will span row(A), so take only the non-zero rows.

MATRIX MULTIPLICATION: (28) ("matrix product")

If $A \in M_{m\times n}(\mathbb{R})$ and $B = [\vec{b}_1 \cdots \vec{b}_k] \in M_{n\times k}(\mathbb{R})$, then the matrix product is:

$$\boxed{AB = [A\vec{b}_1 \cdots A\vec{b}_k] \in M_{m\times k}(\mathbb{R})}$$ A rows dot B cols

for $A = \begin{bmatrix} \vec{r}_1^T \\ \vdots \\ \vec{r}_m^T \end{bmatrix}$: $AB = \begin{bmatrix} \vec{r}_1\cdot\vec{b}_1 & \cdots & \vec{r}_1\cdot\vec{b}_k \\ \vdots & & \vdots \\ \vec{r}_m\cdot\vec{b}_1 & \cdots & \vec{r}_m\cdot\vec{b}_k \end{bmatrix}$ That is, $(AB)_{(i,j)} = \vec{r}_i \cdot \vec{b}_j$

TO BE DEFINED: AB is defined $\Leftrightarrow$ #rows of A equals #cols of B

1. $IA = A = AI$
2. $A(BC) = (AB)C$
3. $A(B+C) = AB + AC$; $(B+C)A = BA + CA$
4. $(cA)B = c(AB) = A(cB)$
5. $(AB)^T = B^TA^T \to (A^k)^T = (A^T)^k$; $\hookrightarrow (A_1 \cdots A_k)^T = A_k^T \cdots A_1^T$

Commutativity?

For every $A, B, C \in M_{m\times n}(\mathbb{R})$, if C commutes with respect to multiplication with A and B, then C commutes with AB and BA.

Complex Matrix

Let $A = [a_{ij}] \in M_{m\times n}(\mathbb{C})$.

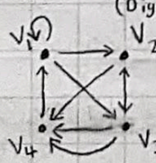
- Conjugate of A: $\overline{A} = [\overline{a_{ij}}]$
- Conjugate transpose: $A^* = \overline{A}^T$

If $A^* = A$, then A is Hermitian

1. $A = (A^*)^*$
2. $(A+B)^* = A^* + B^*$
3. $(\alpha A^*) = \overline{\alpha} A^*$
4. $(AB)^* = B^*A^*$
5. $(A\vec{z})^* = \vec{z}^*A^*$

Application

Directed Graphs (Digraphs w/directed edges)



$\Rightarrow \begin{bmatrix} 1&1&1&0 \\ 0&0&1&1 \\ 0&1&0&2 \\ 1&0&0&0 \end{bmatrix} = A \to$ This is called an Adjacency Matrix

The (i-j)-entry is # of directed edges from V_i to V_j

28.14 # of distinct k-edged paths is given by (i,j)-entry of A^k.

Watch out for double-counting!

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state vector $\vec{S}_k = \begin{bmatrix} S_1^{(k)} \\ \vdots \\ S_n^{(k)} \end{bmatrix}$ Steady state vector $\vec{S} = \begin{bmatrix} S_1 \\ \vdots \\ S_n \end{bmatrix}$

(29) Application (Stochastic Matrices)

Markov Chains: P - stochastic matrix, S - probability vector $\boxed{\vec{S}_{k+1} = P\vec{S}_k}$

1. Determine stochastic matrix + verify regularity
2. Determine initial state vector $\vec{S}_0$ (if required).
3. Solve homogeneous system $(P-I)\vec{S} = \vec{0}$.
4. Choose values for parameters to sum to 1.
5. "By Theorem 29.6, $\vec{S}$ is the steady-state vector."
6. Interpret entries of $\vec{S}$ as needed.

- Probability vector: all entries ≥ 0 and sum to 1.
- Stochastic matrix: all columns are probability vectors.
- State vector: probability vector in Markov Chain.
- Steady-state vector: if $\vec{S}$ satisfies $P\vec{S} = \vec{S}$.

(Every stochastic Matrix has a steady-state vector).
(Finding the SS! $(P-I)\vec{S} = \vec{0}$) (must be square)

- Every state vector is steady for I.
- Regular Matrix: All positive entries in P^k for some k.
  - ↳ also, for any initial state, it converges to steady.
  - ↳ UNIQUE STEADY STATE VECTOR

(30) Matrix Inverses (multiplicative)

Let $A \in M_{n\times n}(\mathbb{R})$. If $\exists B \in M_{n\times n}(\mathbb{R})$ s.t.

$$AB = I = BA \qquad (B = A^{-1})$$

Then A is invertible + B is inverse of A.

$A, B \in M_{n\times n}(\mathbb{R})$ are invertible; then:

1. $(cA)^{-1} = \frac{1}{c}A^{-1}$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^k)^{-1} = (A^{-1})^k$ for $k \in \mathbb{N}$
4. $(A^T)^{-1} = (A^{-1})^T$
5. $(A^{-1})^{-1} = A$

$(A+B)^{-1} \neq A^{-1} + B^{-1}$ necessarily...

Theorem 30.4
$AB = I \iff BA = I$
$AB \Rightarrow \text{rank}(A) = \text{rank}(B)$

Theorem 30.5
$AB = I \wedge AC = I \Rightarrow B = C$

MATRIX INVERSION ALGORITHM

Construct $Ax_1 = \vec{e}_1,\ Ax_2 = \vec{e}_2 \cdots Ax_n = \vec{e}_n$:

$$\left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{n1} & 1 & \cdots & 0 \\ \vdots & & \vdots & & \ddots & \\ a_{1n} & \cdots & a_{nn} & 0 & \cdots & 1 \end{array}\right] \xrightarrow{R_n - xR_m} \left[\begin{array}{ccc|c} 1 & & 0 & A^{-1} \\ & \ddots & & \text{terms!} \\ 0 & & 1 & \end{array}\right]$$

☆☆☆☆☆☆☆☆☆☆☆☆☆☆☆☆☆☆☆☆☆

(31) Properties of Matrix Inverses

Theorem 31.1: $AB = AC \Rightarrow B = C$

Ideas that are logically equivalent: $(A \in M_{n\times n}(\mathbb{R}))$ — Invertible Matrix Theorem

- A is invertible
- $\text{rank}(A) = n$
- RREF of $A = I$
- "Columns" (rows) of A are LI
- $A\vec{x} = \vec{b}$ consistent & unique
- "Columns" (rows) of A span $\mathbb{R}^n$
- A^T is invertible
- $\text{Null}\{A^{(T)}\} = \{\vec{0}\}$

Matrix Transformations ≡ Matrix transformations

Linear transformations

FUNCTION assigns all elements of DOMAIN to unique elements of CODOMAIN (not range)

$f: U \to V$, $u \in U, v \in V$ — "f maps u to v", "v is image of u under f" $\boxed{v = f(u)}$

$f_A: \mathbb{R}^n \to \mathbb{R}^m \ (M_{m\times n}) \equiv f_A(\vec{x}) = A\vec{x},\ \forall \vec{x} \in \mathbb{R}^n$

1. $f_A(\vec{x}+\vec{y}) = f_A(\vec{x}) + f_A(\vec{y})$
2. $f_A(c\vec{x}) = c f_A(\vec{x})$
3. $[F] \in M_{m\times n}(\mathbb{R})$

Linear Transformation

$\forall \vec{x}, \vec{y} \in \mathbb{R}^n$
$\forall s, t \in \mathbb{R}$
$L: \mathbb{R}^n \to \mathbb{R}^m$ IF
$L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$

Linear Operator IF $n = m$

$L: \mathbb{R}^n \to \mathbb{R}^m \Rightarrow$
$[L] = [L(\vec{e}_1) \dots L(\vec{e}_n)] \in \mathbb{R}^m$
defines L: $L(\vec{x}) = [L]\vec{x}$

(STANDARD MATRIX (if L is linear tranformation only))

(32) Solving Stuff ☆☆☆☆☆

We're going to want to look at the lecture notes for this one.

Rotations

Rotate counter-clockwise about origin (by θ):

$$R_\theta(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\vec{x}$$

(33) More Linear Transformations

Stretches and Compressions

$$L(\vec{x}) = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} tx_1 \\ x_2 \end{bmatrix}$$

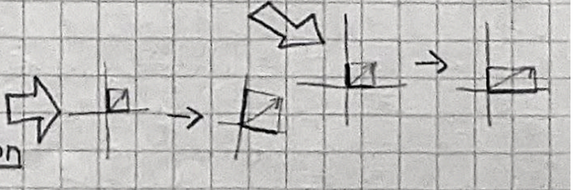
- $t > 1$: L is a Stretch in x_1-direction by a factor of t.
- $0 < t < 1$: L is a Compression of the x_1-direction.

Dilations and Contractions

$$L(\vec{x}) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix} = t\vec{x}$$

— $L(\vec{x})$ is a scalar multiple of $\vec{x}$.

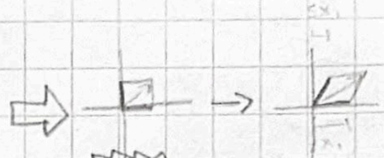
- $t > 1$: L is a dilation
- $0 < t < 1$: L is a contraction



Shears

$$L(\vec{x}) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + s x_2 \\ x_2 \end{bmatrix}$$

- This is a shear in the x_1-direction by a factor of s $(s>0)$



Operations on Linear Transformations

Matrices Equal Theorem: $(L\vec{x} = M\vec{x})$, then $(L = M)$

Definitions 33.2 (functions)
If $M = N$, then
$M(\vec{x}) = N(\vec{x})$,
$[M]\vec{x} = [N]\vec{x}$, and
$[M] = [N]$

Definitions 33.3
For $M, N: \mathbb{R}^n \to \mathbb{R}^m$:
$(M+N)(\vec{x}) = M(\vec{x}) + N(\vec{x})$
$(cM)(\vec{x}) = cM(\vec{x})$
$(M+N)(\vec{x}) = ([M]+[N])(\vec{x})$

Theorem 33.5
If $L, M: \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations and $c \in \mathbb{R}$, then:
$L+M: \mathbb{R}^n \to \mathbb{R}^m$ and $cL: \mathbb{R}^n \to \mathbb{R}^m$ are linear trans:
$[L+M] = [L]+[M]$ and $[cL] = c[L]$

Definition 33.6 ~ Composition
$M \circ L: \mathbb{R}^n \to \mathbb{R}^p$ $(L: \mathbb{R}^n \to \mathbb{R}^m \quad M: \mathbb{R}^m \to \mathbb{R}^p)$
⇩
$M(L(\vec{x}))$ $(\vec{x} \in \mathbb{R}^n)$

Theorem 33.8
If L & M are linear transformations then $M \circ L$ is too and $[M \circ L] = [M][L]$

"Linear Operator" is a Linear Transformation whose codomain == domain

$\mathbb{C}$: complex linear transformations: same as normal! Except complex!

34 Inverse Linear Transformations

- Identity Transformation $Id: \mathbb{R}^n \to \mathbb{R}^n$; $[I] = I$
  $(Id(\vec{x}) = \vec{x} \ \forall \vec{x})$

Translations are Non-linear

- Inverse: $L: \mathbb{R}^n \to \mathbb{R}^n$ is invertible iff there exists $M: \mathbb{R}^n \to \mathbb{R}^n$ such that $M \circ L = Id = L \circ M$. ($M$, then, is the inverse of L). This whole definition is logically equivalent to there being $[M]$ such that $[L]^{-1} = [M]$.
  We then also know that $[L^{-1}] = [L]^{-1}$

- Homogenous Coordinates $(x_1, x_2) \to (x_1, x_2, 1)$
  Add 1 dimension w/value 1 to the end!

Transformation in Homogenous Coordinates:

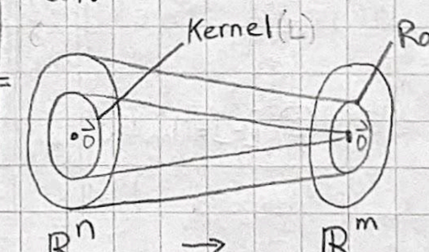
$$A \in M_{n\times n}(\mathbb{R}) \to B \in M_{n+1 \times n+1}(\mathbb{R}) \quad B = \begin{bmatrix} A & \vec{0} \\ \vec{0}^T & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + a \\ x_2 + a \\ 1 \end{bmatrix}$$

35 Kernel + Range (of a LT)

if $L: \mathbb{R}^n \to \mathbb{R}^m$ is LT, then
$Ker(L) = \{\vec{x} \in \mathbb{R} \mid L(\vec{x}) = \vec{0}\}$ $(Ker(L) \subseteq \mathbb{R}^n)$
↳ nullspace, dim = n-rank
and
↳ columnspace, dim = rank
$Range(L) = \{L(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}$ $(Range(L) \subseteq \mathbb{R}^m)$

Theorem 35.5 These ∧ are both subspaces!



ONTO! → surjective
- Range == codomain
- $rank([L]) = m$
- assume $\vec{x} \in \mathbb{R}^m$, then there must be $\vec{y}$ so $L(\vec{y}) = \vec{x}$ (more part of #36)

★ One-to-one (or Injective) iff
$L(\vec{x}_1) = L(\vec{x}_2) \implies \vec{x}_1 = \vec{x}_2$
actually true for all!

If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, L is one-to-one $\iff Ker(L) = \{\vec{0}\}$, AND
L is one-to-one $\iff rank([L]) = n$

36 more hecking definitions

- One-to-One Correspondence (or bijective) if it's one-to-one AND onto.

L is Bijective $\iff$ L is invertible

definition of ONTO:
$\forall \vec{y} \in \mathbb{R}^m, \exists \vec{x} \in \mathbb{R}^n$ such that $[L]\vec{x} = \vec{y}$

$(\dim\{\vec{0}\} = 0)$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

DETERMINANT
$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

ADJUGATE
$adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

If $A \in M_{2\times 2}(\mathbb{R})$
$A(adj\ A) = (\det A)I = (adj\ A)A$

A is invertible $\iff \det A \neq 0$, and then
$A^{-1} = \frac{1}{\det A} adj\ A$

$S \circ P = [S][P]$ ★ $[Adjugate] = [Cofactor]^T$

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If $A \in M_{3\times3}(\mathbb{R})$, $\det(cA) = c^3 \det(A)$

λ eigenvalue and corresponding eigenvector $\vec{v}$, then $A\vec{v} = \lambda\vec{v}$ so $\vec{0} = (A - \lambda I)\vec{v}$
factor out things!

37) Big Determinants $A \in M_{n\times n}(\mathbb{R})$

$\det A = |A|$

$C_{ij} = (i,j)\text{-cofactor} = (-1)^{i+j} \det A(i,j)$ where $A(i,j)$ is A but no ith row or jth column.

$\forall i,\ 1 \le i \le n$

$\det A = a_{i1}C_{i1} + \ldots + a_{in}C_{in}$ → "cofactor expansion of A along the ith row of A"

$\forall j,\ 1 \le j \le n$

$\det A = a_{1j}C_{1j} + \ldots + a_{nj}C_{nj}$ → "cofactor expansion of A along the jth column of A"

DEF 37.7: Let $A = [a_{ij}] \in M_{n\times n}(\mathbb{R})$

① ② cofactor matrix of A is: $\operatorname{cof} A = [C_{ij}] \in M_{n\times n}(\mathbb{R})$

③ adjugate of A is: $\operatorname{adj} A = [C_{ij}]^{T}$

37.9 Theorem $A \in M_{n\times n}(\mathbb{R})$

- $A(\operatorname{adj} A) = (\det A)I = (\operatorname{adj} A)A$
- $A^{-1} = \frac{1}{\det A}\operatorname{adj} A$ iff $\det A \neq 0$

Its All Same For Complexs!

Elementary Row/Column operations

Theorem 37.13 (for $A \in M_{n\times n}(\mathbb{R})$ again)

(1) If A has a row (or column) of zeros, $\det A = 0$

(2) $B = A$ after 1 swap operation, $\det B = -\det A$

(3) $B = A$ after any (including multiples) subtraction or addition operations, $\det B = \det A$

(4) If two rows (or columns) are equal, $\det A = 0$

(5) $B = A$ after multiplying one by c, $\frac{1}{c}\det B = \det A$

38) Determinant Stuff (+ Properties)

$$\begin{vmatrix} x(1-x) & (1+x)(1-x) \\ (1+x)(1-x) & x(1-x) \end{vmatrix} = (1-x)^2 \begin{vmatrix} x & 1+x \\ 1+x & x \end{vmatrix}$$

Triangular: UPPER: every entry below the main diagonal is zero.
LOWER: entries above main diagonal are zero.

Theorem 38.7

If $A = [a_{ij}] \in M_{n\times n}(\mathbb{R})$ is triangular, $\det A = a_{11}a_{22}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$

Theorems:

- ☆ 38.8: $\det(kA) = k^n \det(A)$
- ☆ 38.13: $\det(A^T) = \det(A)$
- ☆ 38.10: $\det(AB) = (\det A)(\det B) = \det(BA)$
- ☆ → $\det(A^k) = (\det A)^k$ $(k > 0)$ $(k \in \mathbb{Z})$
- ☆ 38.11: $\det(A^{-1}) = (\det A)^{-1} = \frac{1}{\det(A)}$

Remember though: $\det(A+B) \neq \det A + \det B$ in general

39) Polynomial Interpolation and Determinants with Area

Polynomial Interpolation: Finding a polynomial satisfying points

① get n points; call them (x_i, y_i)

② $P(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$

③ Set up equations, leading to:

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}}_{\text{Vandermonde Matrix}} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Let A be a Vandermonde Matrix. Then:

$\det A = \prod_{1 \le i < j \le n} (x_j - x_i)$

IMPORTANT

39.2 If $x_1 \cdots x_n$ are distinct, then a unique polynomial P, $\deg(P) = n-1$ exists. for the points

solving that for all the as...

① get $\det A$. (with $\prod$ ...)

② get $\operatorname{adj} A$. (LOTS OF WORK w/ cofactors)

③ $\begin{bmatrix} a_0 \\ \vdots \\ a_{n-1} \end{bmatrix} = \frac{1}{\det A}\operatorname{adj} A \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$\underbrace{A\begin{bmatrix} a_1 \\ a_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_n \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_n \end{bmatrix} = A^{-1}\begin{bmatrix} y_1 \\ y_n \end{bmatrix}}$

AREA: The area of a parallelogram created by $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ is given by:

$\text{AREA} = |\det[\vec{u}\ \vec{v}]|$ absolute value. ☺

found using realizations in $\mathbb{R}^3$:

Stretching/skewing shapes with given areas, we have: $A_{new} = |\det[L]|A_{shape}$, where $[L]$ is the linear transformations
area!

40 Eigen-hecking-Things (horray.)

$\lambda \in \mathbb{C}$

☆ B^2+3B multiply by $\lambda+3$

VOLUMES

$V = |\vec{u}\cdot(\vec{v}\times\vec{w})|$

$\Downarrow$

$V = |\det[\vec{u}\ \vec{v}\ \vec{w}]|$

For new special-special shapes and such, we have:

$V_{new} = |\det[L]|\, V_{old\_shape}$

eg $V_{sphere} = \frac{4}{3}\pi$

X_2 stretch by 2, and X_3 stretch by 3 gives

$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ so

$V_{ellipsoid} = |1\cdot 2\cdot 3|\,\frac{4}{3}\pi$

40.1 $A \in M_{n\times n}(\mathbb{R})$ and $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$:
An eigenvalue λ of A satisfies $A\vec{x} = \lambda\vec{x}$.
$\vec{x}$ is then an eigenvector of A corresponding to λ

Finding/using eigenvalues

40.4 λ is an eigenvalue of $A \iff \det(A-\lambda I) = 0$
all $\vec{x}$ corresponding to λ are solutions to $(A-\lambda I)\vec{x} = \vec{0}$

40.5 Characteristic Polynomial of A is
$C_A(\lambda) = \det(A-\lambda I)$ so λ is an eigenvalue of A if and only if $C_A(\lambda) = 0$ $\hookrightarrow \deg(C_A) = n$!

EigenSpace: $\text{null}(A-\lambda I)$

41 OTHER EIGENTHINGS

41.2 set containing all the eigenvectors corresponding to λ together with the $\vec{0}$ is called the eigenspace of A corresponding to λ, and is denoted by E_λ:

$$E_\lambda(A) = \text{Null}(A-\lambda I)$$

41.4 Algebraic Multiplicity of λ, called a_λ, is # of times λ appears as a root of $C_A(\lambda)$.

41.5 Geometric Multiplicity of λ, g_λ, is $g_\lambda = \dim(E_\lambda(A))$

41.6 $1 \leq g_\lambda \leq a_\lambda \leq n$

→ invertible $\iff$ no element on main diagonal is zero!

Triangular Matrix: eigenvalues of A are the entries on the main diagonal of A!

A real matrix can have non-real eigenvalues

If $\vec{x}$ is eigenvector of real matrix A corresponding to λ, then $\bar{\vec{x}}$ is an eigenvector of A corresponding to $\bar{\lambda}$.

42 Diagonalization

Diagonal Matrix: Both upper and lower triangular: $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

$D = \text{diag}(d_{11}, \cdots d_{nn})$ $D_1 = \text{diag}(6, 0, -4)$

LEMMA 42.4 If $D = \text{diag}(d_{11}, \cdots d_{nn})$ and $E = \text{diag}(e_{11}, \cdots e_{nn})$ then:
(1) $D+E = \text{diag}(d_{11}+e_{11}, \cdots, d_{nn}+e_{nn})$
(2) $DE = \text{diag}(d_{11}e_{11}, \cdots, d_{nn}e_{nn}) = ED$
(3) $D^k = \text{diag}(d_{11}^k, \cdots, d_{nn}^k)$ if none of $d_{ii} = 0$, $(\forall k \in \mathbb{Z})$
(that is, if D is invertible)

42.5 $n\times n$ matrix A is diagonalizable if there exists $n\times n$ invertible matrix P and $n\times n$ diagonal matrix D such that $P^{-1}AP = D$, then "P diagonalizes A to D"

$A = PDP^{-1}$

42.6 So if $P^{-1}AP = D$, then:
(1) $\det A = \det D$ $PDP^{-1} = A$
(2) A and D have same eigenvalues
(3) $\text{rank}(A) = \text{rank}(D)$
(4) $\text{tr}(A) = \text{tr}(B)$ where the trace is: $\text{tr}(A) = a_{11} + \ldots + a_{nn}$

42.7 So then A and D are similar.

So A is diagonalizable iff it is similar to a diagonal matrix.

LEMMA 42.8 Let $\lambda_1, \cdots \lambda_n$ be distinct eigenvalues of $A \in M_{n\times n}(\mathbb{R})$. Let B_i is a basis for each eigenspace $E_{\lambda_i}(A)$, $i = 1, \ldots k$, then $B = B_1 \cup B_2 \cup \ldots \cup B_k$ is linearly independent.

42.9 DIAGONALIZATION THEOREM
$A_{n\times n}$ is diagonalizable iff there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

$a_{\lambda_k} = g_{\lambda_k}$ (all k) $\iff A$ is diagonalizable

n distinct eigenvalues $\implies A$ is diagonalizable

Given A: jth column of P contains jth vector from basis of eigenvectors, and j^{th} column of D has corresponding eigenvalue in (j,j)-entry.

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A is invertible $\iff \lambda_i \neq 0 \ \forall i \in \mathbb{N}$

43 Powers of Matrices

Let $A \in M_{n\times n}(\mathbb{R})$ be diagonalizable. Then $P^{-1}AP = D$ for some invertible $P \in M_{n\times n}(\mathbb{R})$ and $n\times n$ diagonal matrix D. Then, for any $k \in \mathbb{N}$, we have: $A^k = PD^kP^{-1}$

$C_A(\lambda) = \det(A - \lambda I) = (-1)^n(\lambda-\lambda_1)^{a_{\lambda_1}} \cdots (\lambda - \lambda_k)^{a_{\lambda_k}}$

take $\lambda = 0$; then: $\det A = \prod_{i=1}^{k} \lambda_i^{a_{\lambda_i}}$

Trace: $\operatorname{tr} A = \sum_{i=1}^{k} \lambda_i a_{\lambda_i}$

44 Vector Space

Definition 44.1

A set $\mathbb{V}$ with an operation of addition ($\vec{x}+\vec{y}$) and an operation of multiplication ($c\vec{x}$, $c \in \mathbb{R}$) is called a Vector Space over $\mathbb{R}$ if for every $\vec{v}, \vec{x}, \vec{y} \in \mathbb{V}$ and for every $c, d \in \mathbb{R}$:

- V1: $\vec{x}+\vec{y} \in \mathbb{V}$
- V2: $\vec{x}+\vec{y} = \vec{y}+\vec{x}$
- V3: $(\vec{x}+\vec{y})+\vec{v} = \vec{x}+(\vec{v}+\vec{y})$
- V4: $\exists \vec{0} \in \mathbb{V}$, "zero vector", s.t. $\vec{x}+\vec{0} = \vec{x}$
- V5: for all $\vec{x}$, exists $-\vec{x} \in \mathbb{V}$ so $\vec{x}+(-\vec{x}) = \vec{0}$
- V6: $c\vec{x} \in \mathbb{V}$
- V7: $c(d\vec{x}) = (cd)\vec{x}$
- V8: $(c+d)\vec{x} = c\vec{x}+d\vec{x}$
- V9: $c(\vec{x}+\vec{y}) = c\vec{x}+c\vec{y}$
- V10: $1\vec{x} = \vec{x}$

EXAMPLES $\rightarrow$ 44.2

VECTOR SPACES: $\mathbb{R}^n$ (44.2), Subspaces of $\mathbb{R}^n$ (44.2), $\mathbb{L}(L: \mathbb{R}^n \to \mathbb{R}^m)$ (44.3), $\mathcal{F}(a,b)$ of $f: (a,b) \to \mathbb{R}$ (44.4, all functions), $C(a,b)$ of all continuous functions (44.4), $C^1(a,b)$ of all differentiable functions (44.4)

Discontinuous functions are not vector spaces

Definition 44.6 **SPAN**

Let $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in vectorspace V. Span of B is $\operatorname{Span} B = \{c_1\vec{v}_1 + \dots + c_k\vec{v}_k \mid c_1, \dots c_k \in \mathbb{R}\}$

- Span$B$ is spanned by B
- B is a spanning set for Span$B$

Example 44.14 We have: $\dim(M_{m\times n}(\mathbb{R})) = mn$

$P(\mathbb{R}) \rightarrow$ all real polynomials

$P_n(\mathbb{R}) \rightarrow$ polynomials with degree less or equal to n.

Definition 44.7 **LI**

Linear independence: same as before!

Definition 44.8 **SUBSPACE**

$S \subseteq V$ is a subspace of $\mathbb{V}$ if D1–D10 of Definition 44.1 hold ($\vec{x}, \vec{y}, \vec{z} \in S$.)

- $\{\vec{0}\}$ is the "Trivial Subspace" of V, and V is a subspace of V.

Theorem 44.9 **SUBSPACE TEST**

[1] $S \subseteq V$: S nonempty subset of V. [2] $\vec{0} \in \operatorname{span} S$?

[3] $\forall \vec{x}, \vec{y} \in S \ \forall c \in \mathbb{R}$, $\vec{x}+\vec{y} \in S \wedge c\vec{x} \in S$

then S is a subspace of V

Definition 44.10 **BASIS**

S subspace of V. Let $B \subseteq S$. If B is LI and $S = \operatorname{Span} B$, B is a basis for S. (if $S = \{\vec{0}\}$, $B = \emptyset$ is basis)

Definition 44.11 **DIMENSION**

Dimension of a subspace S of V, $\dim(S)$, is # of vectors in any basis for S.

45 Polynomials

For $p(x), q(x) \in P_n(\mathbb{R}) \rightarrow$ $p = a_0 + a_1x^1 + \dots + a_nx^n$, $q = b_0 + b_1x + \dots + b_nx^n$

- $p(x) = q(x)$ iff $a_i = b_i \ \forall i \in \mathbb{Z}, 0 \le i \le n$
- $(p+q)(x) = p(x)+q(x) = (a_0+b_0)+(a_1+b_1)x^1 + \dots + (a_n+b_n)x^n$
- $(kp)(x) = kp(x) = ka_0 + ka_1x + \dots + ka_nx^n$, $k \in \mathbb{R}$

$P_n(\mathbb{R})$ is a vector space.

Standard basis of $P_n(\mathbb{R})$: $B = \{1, x, \dots, x^n\} \subseteq P_n(\mathbb{R})$

$\dim(P_n(\mathbb{R})) = n+1$ — Example 45.3

The standard BASIS OF $P(\mathbb{R})$ is $B = \{1, x, x^2, \dots\}$ and it is infinite-demensional!

Showing it is a Basis: Having the correct number of elements and being linearly independent is all you need!

Finding a Basis of a subspace: RREF, and elements with leading entries are yours! let $ax+b$, or whatever, be the factor you don't know.

46 vector-wector

$x \oplus y = xy$

$c \odot x = x^c$

and Complex-Womplex

Theorem 46.2

If V is vector space, then for $\vec{x} \in V$:

(1) $0\vec{x} = \vec{0}$

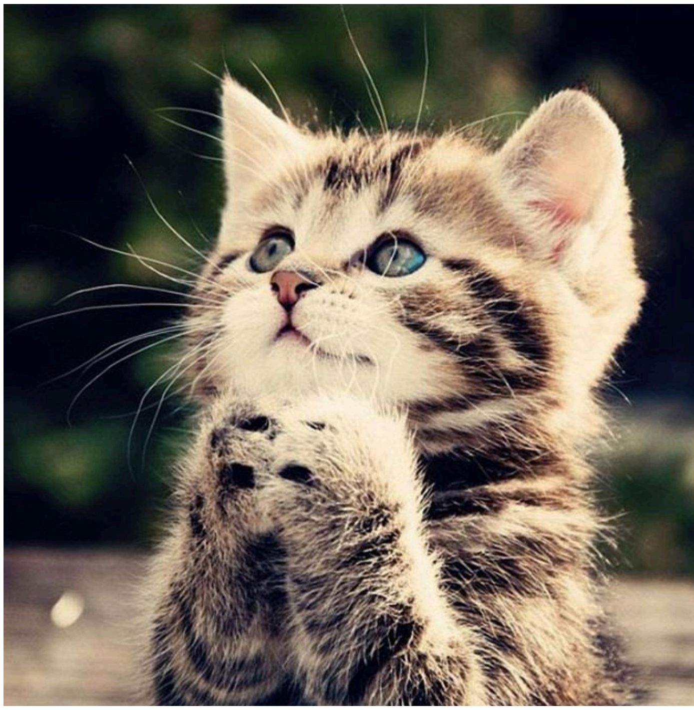
(2) $-\vec{x} = (-1)\vec{x}$

$\mathbb{C}^n$ is vector space over $\mathbb{C}$ — $\dim(\mathbb{C}^n) = n$

$M_{m\times n}(\mathbb{C})$ is vector space over $\mathbb{C}$ — $\dim(M_{m\times n}(\mathbb{C})) = mn$

$P_n(\mathbb{C})$ is vector space over $\mathbb{C}$ — $\dim(P_n(\mathbb{C})) = n+1$

Congrats, Josiah! You finished your first Uni course!



the cute cat is praying for good marks

the cute dog is ready for the final

the smiling quokka is stress-eating